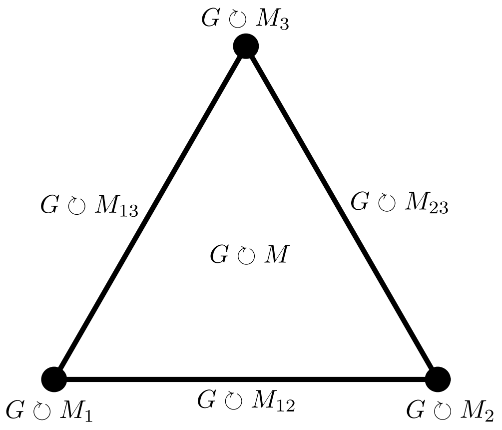
Diplomarbeit

Equivariant ad theories

vorgelegt von Steffen Wittkamp
geb. am 10.09.1983 in Essen



18.02.2010

Lehrstuhl Topologie
Fakultät für Mathematik
der Ruhr-Universität Bochum

Betreuer: Prof. Dr. Gerd Laures

Contents

1 Introduction

In [Qui95] Frank Quinn introduces a machinery for constructing spectra from "bordism-type theories". Examples of bordism-type theories arise from manifold n-ads, Poincaré n-ads and from Ranicki's symmetric and quadratic Poncaré n-ads. In [LM09], Gerd Laures and James E. McClure define a structure with stronger axioms, which they call an *ad theory*.

An ad theory consists of a target category $\mathcal{A}$ and the ads, which are certain functors from the categories of oriented cells of a ball complex into the target category. These ads have to fulfill the axioms of an ad theory. For example, the oriented compact topological manifolds with boundary form the target category of the ad theory of oriented topological bordism. A bordism between two closed manifolds is a functor F from the category of the oriented cells of the ball complex I (the complex consisting of one 1-cell whose boundary is the disjoint union of two 0-cells) to manifolds with boundary which has certain properties: It shifts dimensions by a fixed value, it is compatible with taking boundaries and it somehow respects orientations.

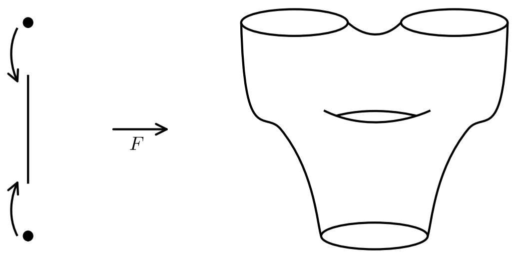


Figure 1: A bordism as an I-ad.

Such bordisms are the I-ads of this ad theory. If K is an arbitrary ball complex, then the K-ads can be seen as a generalization of the concept of bordism. For example if K is Δ^n then we speak of n-ads. A morphism of ad theories is a functor between the target categories that preserves ads.

To an ad theory one assigns its Ω-spectrum Q (this is Quinn's spectrum construction) and this defines a functor from the category of ad theories to the category of Ω-spectra. The spaces Q_k of the spectrum Q are the realizations of the semisimplicial sets defined by the n-ads of degree k. Furthermore, one can assign to an ad theory its bordism groups and a cohomology theory representing this bordism groups which is naturally isomorphic to the cohomology theory associated to Q.

The construction of Q only depends on the target category and not on a bundle theory like the Pontryagin-Thom construction. Another advantage is that there is no choice of transversal intersections needed: The idea is, that by looking at the Δ^n-ads one only considers those simplices that define transversal intersections when restricted to the faces of Δ^n.

Laures and McClure prove that all the standard examples of bordism type theories fulfill the axioms of an ad theory and they construct a functor from ad theories to the category of symmetric spectra which is weakly equivalent to Quinn's construction. Furthermore they show that the symmetric spectrum is a strictly associative ring spectrum if the ad theory is multiplicative (see [LM09, Definition (16.4)]). Multiplicativity is analogous to the existence of Cartesian products for topological manifolds and its compatibility with dimension, reversion of the orientation and bordism.

The aim of this work is to answer the question, what an equivariant ad theory should be and how equivariant ad theories can be constructed. In particular we want to get some kind of equivariant spectra and equivariant (co)homology theories from such constructions.

For manifolds groups appear, if one generalizes to G-manifolds and G-bordism. Guided by this example, the first step is to construct ad theories with target categories consisting of G-objects.

It now turns out to be useful to adopt the point of view on equivariant (co)homology theories that is taken in the work of Wolfgang Lück and others (see [DL98], [KL05] and [Lüc05]). There equivariant (co)homology theories consist of G-(co)homology theories which are linked by a so-called induction structure. In particular one gets G-(co)homology theories from functors of the orbit category $\mathrm{or}(G)$ to (Ω)-spectra and equivariant (co)homology theories from functors from a category of small groupoids to (Ω)-spectra which take equivalences of groupoids to weak equivalences of spectra. The category of groupoids has to contain all transport groupoids of homogeneous spaces of the form G/H for subgroups H of a group G.

Because of the functorial flavor of the definition of ad theories and the fact that an equivalence of target categories induces isomorphisms of the bordism groups and therefore weak equivalences between the associated spectra, we are lead to a more general question: Suppose given an ad theory with target category $\mathcal{A}$, is it possible to construct a new ad theory whose target category is the functor category of functors $\mathcal{C} \to \mathcal{A}$, where $\mathcal{C}$ is a small category? In particular $\mathcal{C}$ should be allowed to be the transport groupoid of a homogeneous space of the form G/H. This would define a contravariant functor from groupoids to Ω-spectra as above and would allow us to apply the construction of an equivariant cohomology theory to it.

The first we have to do is to restrict ourselves to those small categories $\mathcal{C}$ that have exactly one isomorphism class of objects. This ensures that we have a well-defined notion of dimension for the objects of the functor category. The transport groupoids of the G-sets G/H all have this property, in fact they are equivalent to the category of the group H.

Then we discover two new properties of ad theories, a functoriality property of the gluing constructions and a functoriality of the cylinder constructions. See Section 3.2 for details. The main result (see Theorem 3.3.1) of this work is, that these properties are sufficient to construct a new ad theory whose target category is $\mathcal{A}^{\mathcal{C}}$.

The construction of this new ad theory is functorial in the ad theory and in $\mathcal{C}$. We observe that the new ad theory is multiplicative if the old ad theory was multiplicative. We generalize this construction to functors to a larger version of the original target category which contains more morphisms. This is necessary, because for example the target category of oriented topological bordism contains only inclusions, but we need at least orientation-preserving homeomorphisms to be able to describe non-trivial G-actions. We also examine the case of a full subcategory of the new target category of functors, to be able to construct ad theories of manifolds with continuous G-action and with restricted isotropy.

Having done all this we can apply the construction of equivariant cohomology theories to such an ad theory with functorial gluing and cylinder constructions. The result is an equivariant cohomology theory that represents the bordism groups of the new ad theories with values in $\mathcal{A}^{\mathcal{G}}$, where $\mathcal{G}$ is the category of the group G.

We prove that all the standard examples of ad theories have these functoriality properties of the gluing and cylinder constructions. The bordism groups of the new ad theory with values in topological oriented G-manifolds are the usual G-bordism groups, which justifies our construction. Thus we get an equivariant cohomology theory representing equivariant bordism of topological manifolds. We also get the G-homology theory for bordism of oriented topological G-manifolds. It is remarkable, that the associated Quinn spectrum really represents G-bordism, because this is generally not true for the Thom spectrum due to a failure of transversality in the equivariant situation.

We also get spectra and equivariant theories for geometric, symmetric and quadratic Poincaré ad theories and we conclude with some remarks on another possible construction of equivariant ad theories for symmetric and quadratic Poincaré ad theories.

Outline of this work: First we introduce ad theories and the main results of the work of Gerd Laures and James E. McClure in Section 2. Examples are deferred to Section 5. Secondly, we define the new functoriality conditions and construct the new ad theories whose target categories are diagram categories. This is done in Section 3. The generalizations mentioned above are also carried out there.

The notions of or(G)-spectra, G-(co)homology theories and equivariant (co)homology theories are introduced at the beginning of Section 4. We then apply the methods of the works of Lück and others (see [Lüc05], [DL98] and [KL05]) to ad theories with functorial gluing and cylinder constructions to get the equivariant cohomology theories we wanted.

In Section 5 we investigate the standard examples and show that they all have functorial gluing and cylinder constructions. We examine some of the new ad theories and equivariant cohomology theories we get.

The last chapter is an appendix which contains basic definitions and results we will need throughout this work. We refer to it when necessary.

Acknowledgements. First, I would like to thank Prof. Dr. Gerd Laures for the subject of this work and his support. Secondly I want to say thank you to all the members of the topology chair at the Ruhr-Universität Bochum for all the helpful discussions and the pleasant atmosphere. In particular I am grateful to Dr. Markus Szymik. I appreciate the financial support from the Studienstiftung des deutschen Volkes (German scholarship foundation) during my studies.

2 Ad theories

Here we want to introduce ad theories which are the main subject of this work. We cite the definitions from [LM09] and occasionally we add small examples or remarks, which we will use later. We also present the main results of [LM09]. Examples for ad theories are deferred to Section 5, where we also show that all the standard examples have the functorial gluing and cylinder properties, which we introduce in Section 3.2.

2.1 Axioms

Definition 2.1.1. A *category with involution* is a category together with an endofunctor i which satisfies $i \circ i = \mathrm{id}$.

Example 2.1.2. The set $\mathbb{Z}$ is partial ordered and therefore it defines a category. With i being the identity functor it gives an example for a category with involution.

Definition 2.1.3. A $\mathbb{Z}$*-graded category* is a category $\mathcal{A}$ with involution i together with two functors $d\colon \mathcal{A} \to \mathbb{Z}$ (called *dimension functor*) and $\emptyset\colon \mathbb{Z} \to \mathcal{A}$ which strictly commute with i and fulfill $d\emptyset = \mathrm{id}$. A functor between $\mathbb{Z}$-graded categories is called a k*-morphism* if it strictly commutes with i and $\emptyset$ and decreases the dimension by k.

Note that because of d being a functor there can not exist any morphism $A \to B$ in a $\mathbb{Z}$-graded category if $d(A) > d(B)$. We often write $\emptyset_n$ for $\emptyset(n)$ and sometimes $d_{\mathcal{A}}$, $i_{\mathcal{A}}$ and $\emptyset_{\mathcal{A}}$ to clarify to which $\mathbb{Z}$-graded category the dimension functor, the involution or $\emptyset$ belong.

Example 2.1.4. A chain complex C defines a $\mathbb{Z}$-graded category as follows: The objects of dimension n are the elements of C_n and in addition to identities we have a unique morphism from every object to every object of higher dimension. The involution is multiplication with -1 and $\emptyset_n$ is the element $0 \in C_n$. Then the boundary map is an example for a 1-morphism.

Example 2.1.5. We denote by $\mathcal{A}_{STop}$ the category whose objects are the compact oriented topological manifolds with boundary together with an empty manifold $\emptyset_n$ for every n; the dimension-preserving morphisms are the orientation-preserving inclusions of manifolds with boundary (in particular the image of the boundary has to be a subset of the boundary) and the dimension-increasing morphisms are the inclusions with image in the boundary. The involution reverses the orientation. This defines a $\mathbb{Z}$-graded category and taking boundary defines an endofunctor, which is a 1-morphism.

Example 2.1.6. Let $\mathcal{D}_{STop}$ be the $\mathbb{Z}$-graded category of compact oriented topological manifolds with boundary together with an empty manifold $\emptyset_n$ for every dimension n, whose dimension-increasing morphisms are the embeddings of topological manifolds with boundary (with image in the boundary) and whose dimension-preserving morphisms are orientation-preserving embeddings. Involution is again reversion of orientations. Then $\mathcal{A}_{STop}$ is a subcategory of $\mathcal{D}_{STop}$ that respects the $\mathbb{Z}$-graded structure in the sense of the following definition:

Definition 2.1.7. Let $\mathcal{A}$ and $\mathcal{D}$ be $\mathbb{Z}$-graded categories and let $\mathcal{A}$ be a subcategory of $\mathcal{D}$. Then we call $\mathcal{A}$ a *$\mathbb{Z}$-graded subcategory* of $\mathcal{D}$ if the inclusion functor $\mathcal{A} \to \mathcal{D}$ is an inclusion of $\mathbb{Z}$-graded categories, that is it is a 0-morphism. This is equivalent to $\emptyset$ and the involution of $\mathcal{D}$ restricting to that of $\mathcal{A}$.

We proceed with the example of a $\mathbb{Z}$-graded category that will be the source of the models for ads, the category of cells of a ball complex. Basic definitions and properties of ball complexes and references to the literature are given in part 6.2 of the Appendix. One can also think of a ball complex as a finite regular cell complex together with an underlying piecewise linear structure.

Example 2.1.8. Let L be a subcomplex of a ball complex K. Let $\mathcal{C}\mathrm{ell}(K, L)$ be the $\mathbb{Z}$-graded category whose objects are the oriented closed cells (σ, o) which are not in L and additionally an object $\emptyset_n$ for every $n \in \mathbb{Z}$. The morphisms are the identities, the inclusions of cells into cells of higher dimension without any requirements on the orientations, and a morphism from $\emptyset_n$ to each object of higher dimension. The involution is the reversion of the orientations. We often write $\mathcal{C}\mathrm{ell}(K)$ for $\mathcal{C}\mathrm{ell}(K, \emptyset)$.

Every morphism of pairs of ball complexes induces a 0-morphism between the $\mathcal{C}$ell-categories. We are furthermore interested in abstract morphisms which are not necessarily induced by maps of pairs. Recall the definition of incidence numbers from Section 6.2 of the Appendix.

Definition 2.1.9. Let

$$\theta \colon \mathcal{C}\mathrm{ell}(K_1, L_1) \to \mathcal{C}\mathrm{ell}(K_2, L_2)$$

be a k-morphism.

(i) Then θ is called *incidence-compatible* if

$$[(\sigma, o), (\sigma', o')] = (-1)^k [\theta(\sigma, o), \theta(\sigma', o')]$$

for all pairs $((\sigma, o), (\sigma', o'))$ of oriented cells in $\mathcal{C}\mathrm{ell}(K_1, L_1)$.

(ii) Let $\mathcal{A}$ be a $\mathbb{Z}$-graded category and $F \colon \mathcal{C}\mathrm{ell}(K_2, L_2) \to \mathcal{A}$ an l-morphism, then the composition $i^{kl} \circ F \circ \theta$ is denoted by $\theta^* F \colon \mathcal{C}\mathrm{ell}(K_1, L_1) \to \mathcal{A}$.

Definition 2.1.10. Let $\mathcal{A}$ be a $\mathbb{Z}$-graded category and let L be a subcomplex of a ball complex K. Then a *pre (K, L)-ad of degree k with values in $\mathcal{A}$* is a k-morphism $\mathcal{C}\mathrm{ell}(K, L) \to \mathcal{A}$.

We often only speak of pre (K, L)-ads if it is clear what $\mathcal{A}$ is. The pre (K, L)-ads of degree k form a set which will be denoted by $\mathrm{pre}^k(K, L)$. If L is the empty ball complex, then the set of pre (K, L)-ads of degree k is denoted by $\mathrm{pre}^k(K)$. Note that pre^k defines a functor from (pairs of) ball complexes to sets.

Definition 2.1.11. An *ad theory* consists of

- a $\mathbb{Z}$-graded category $\mathcal{A}$ (called *target category*)
- for each k a subfunctor ad^k of pre^k (the subset $\mathrm{ad}^k(K,L)$ of $\mathrm{pre}^k(K,L)$ is called the set of (K,L)-*ads of degree* k) such that the following axioms hold:
  (a) For each pair (K,L) of ball complexes the equality
  $$\mathrm{ad}^k(K,L) = \mathrm{pre}^k(K,L) \cap \mathrm{ad}^k(K)$$
  holds.
  (b) For each ball complex K, the element of $\mathrm{pre}^k(K)$ which takes every object of $\mathcal{C}\mathrm{ell}(K)$ to $\emptyset$ is a K-ad, called the *trivial K-ad* of degree k.
  (c) The involution $i_{\mathcal{A}}$ takes K-ads to K-ads.
  (d) A pre K-ad that is isomorphic to a K-ad is a K-ad.
  (e) A pre K-ad is a K-ad if it restricts to a σ-ad for each closed cell σ of K.
  (f) (Reindexing) If
  $$\theta\colon \mathcal{C}\mathrm{ell}(K_1,L_1) \to \mathcal{C}\mathrm{ell}(K_2,L_2)$$
  is an incidence-compatible k-isomorphism of $\mathbb{Z}$-graded categories, then the induced bijection
  $$\theta^*\colon \mathrm{pre}^l(K_2,L_2) \to \mathrm{pre}^{l+k}(K_1,L_1)$$
  restricts to a bijection
  $$\theta^*\colon \mathrm{ad}^l(K_2,L_2) \to \mathrm{ad}^{l+k}(K_1,L_1).$$
  (g) (Gluing) For each subdivision K' of K and each K'-ad F there is a K-ad which agrees with F on each residual subcomplex.
  (h) (Cylinder) There is a natural transformation
  $$J\colon \mathrm{ad}^k(-) \to \mathrm{ad}^k(- \times I)$$
  such that for $F \in \mathrm{ad}(K)$ the restrictions of $J(F)$ to $K \times 0$ and $K \times 1$ are both equal to F and J takes trivial ads to trivial ads. (Note, that here $K \times I$ has the canonical ball complex structure of the product.)

A *morphism* (resp., *equivalence*) of ad theories is a functor (resp., equivalence) of the target categories that takes ads to ads. Often an ad theory with target category $\mathcal{A}$ will be denoted by $\mathrm{ad}_{\mathcal{A}}$.

2.2 The bordism groups of an ad theory

Throughout this section we fix an ad theory with target category $\mathcal{A}$.

Definition 2.2.1. Let $*$ denote the space consisting of one point with its ball complex structure (given by one cell of dimension 0). Two elements of $\mathrm{ad}^k(*)$ are *bordant* if there exists an I-ad which restricts to the given ads at the ends.

Remark 2.2.2. This defines an equivalence relation: Reflexivity is a consequence of the cylinder axiom, symmetry follows from the reindexing axiom and transitivity from the gluing axiom.

Definition 2.2.3. The set of bordism classes in $\mathrm{ad}^{-k}(*)$ is denoted by Ω_k. The bordism class of an ad F is denoted by $[F]$. We sometimes write $\Omega_k(\mathrm{ad}_{\mathcal{A}})$ to specify the ad theory whose bordism group we mean.

Next it is shown in [LM09] that the sets Ω_k have an abelian group structure. We give a short description of it: The isomorphism of categories

$$\mathcal{C}\mathrm{ell}(I, \{0, 1\}) \to \mathcal{C}\mathrm{ell}(*)$$

defines an incidence-compatible 1-morphism which will be denoted by κ. By the reindexing axiom the map

$$\kappa^*\colon \mathrm{ad}^k(*) \to \mathrm{ad}^{k+1}(I, \{0, 1\})$$

is a bijection.

Now let $F, G \in \mathrm{ad}^k(*)$. Performing the following steps one by one (see Figure 2), will give a construction of the sum:

(i) Apply κ^*. Then $\kappa^* F$ and $\kappa^* G$ define $(I, \{0, 1\})$-ads of degree $k + 1$.

(ii) Let F' and G' be the extensions of these ads to I-ads by defining them to be the trivial ad on the boundary of I. This defines ads by axiom (a) of Definition 2.1.11.

(iii) Apply the cylinder axiom to get $J(F')$ and $J(G')$. Note that these ads restrict to the trivial ads on the left and right edges (see Figure 2).

(iv) Use the gluing axiom to glue these ads together along these edges. This defines an ad on the ball complex one gets by gluing together two copies of $I \times I$ along one edge. Let M' denote this ball complex.

(v) Use gluing to get an ad on the subdivision of M' shown in the Figure. Denote this subdivision by M.

(vi) Take the restriction of this ad on M to the upper edge.

(vii) It is the trivial ad on the boundaries, so it defines an $(I, \{0, 1\})$-ad.

(viii) Apply $(\kappa^{-1})^*$ to get a $*$-ad.

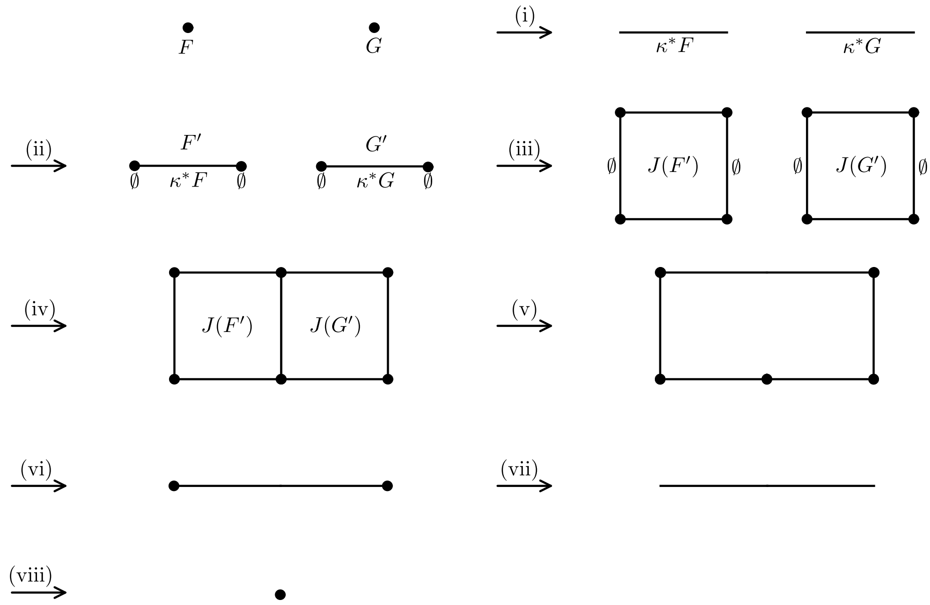


Figure 2: Construction of the sum.

This construction up to step (v) proves the following lemma:

Lemma 2.2.4 ([LM09, Lemma (4.4)])**.** *For $F, G \in \mathrm{ad}^k(*)$, there exists an ad H of degree $k+1$ on M such that it restricts to F' and G' on the two lower edges and that it is the trivial ad on the vertical edges of M.* □

Definition 2.2.5. Let $F, G \in \mathrm{ad}^k(*)$ and H an ad on M as in Lemma 2.2.4. Apply steps (vi) to (viii) and define $[F] + [G]$ to be the resulting $*$-ad of degree k.

The operation $+$ is well-defined: This can be seen by looking at page 8 and Figure 3 of [LM09].

Proposition 2.2.6 ([LM09, Proposition (4.7)])**.** *The operation $+$ makes Ω_k an abelian group.* □

Remark 2.2.7 ([LM09, Remark (4.8)]). An equivalence of ad theories induces an isomorphism of the bordism groups.

This remark will become important for us in Section 4, where we want to get equivariant theories from ad theories.

2.3 The cohomology theory of an ad theory

We continue by describing the cohomology theory associated to an ad theory.

Definition 2.3.1. Let K be a ball complex and L a subcomplex. Two ads $F, G \in \mathrm{ad}^k(K, L)$ are *bordant* if there exists a $(K \times I, L \times I)$-ad which restricts to F on $K \times 0$

and to G on $K \times 1$. As for bordism groups this is an equivalence relation and we write $T^k(K, L)$ for the set of bordism classes in $\mathrm{ad}^k(K, L)$ and the bordism class of F will be denoted by $[F]$.

Remark 2.3.2. $T^k(*)$ is the same as Ω_{-k}.

The sets $T^k(K, L)$ have an abelian group structure that can be defined similar to the case of bordism groups: There is an incidence-compatible isomorphism

$$\kappa\colon \mathcal{C}\mathrm{ell}(I \times K, (\{0,1\} \times K) \cup (I \times L)) \to \mathcal{C}\mathrm{ell}(K, L),$$

which takes $I \times (\sigma, o)$, where I has the standard orientation, to (σ, o), so κ^* induces a bijection of ads by the reindexing axiom. If M and M' are the ball complexes we used above to define the addition of the bordism groups, then there is a generalized version of the construction and therefore of Lemma 2.2.4. This generalized version shows that, given $F, G \in \mathrm{ad}^k(K, L)$, there is an $H \in \mathrm{ad}^{k+1}(M \times K, M \times L)$ which restricts to the analogues of F' and G' on the products of the lower edges with K and is trivial on products of the vertical edges of M with K.

Again, one can take such an ad H and apply the analogues of the steps (vi) to (viii). Then $[F] + [G]$ is defined to be the resulting (K, L)-ad and one uses the same arguments as for the bordism groups to show that this gives a well-defined abelian group structure on $T^k(K, L)$. See sections 4 and 12 of [LM09] for further details.

Next it is shown in [LM09] that this defines a homotopy functor. Recall from Section 6.2, that Bi denotes the category of pairs of ball complexes and Bh is the category consisting of the same objects whose morphisms are homotopy classes of continuous maps of pairs.

Proposition 2.3.3 ([LM09, Proposition (12.5)]). *For every k the functor $T^k\colon Bi \to Ab$ factors uniquely through Bh.* □

The proof uses elementary expansions (see Definition (12.6) and Lemma (12.7) in [LM09] and [BRS76, p. 5]) to show that the assumptions of Proposition I.6.1 and Theorem I.5.1 of [BRS76] are fulfilled.

Then one wants to show that the functors T^k define a cohomology theory. First one observes, that excision is a direct consequence of the reindexing axiom.

One constructs a connecting homomorphism as follows: First one can proof (see [LM09, Lemma (12.8)]), that κ^* induces an isomorphism

$$T^k(L) \to T^{k+1}(I \times L, \{0,1\} \times L)$$

of abelian groups. Then one uses that excision produces an isomorphism

$$T^k(I \times L, \{0,1\} \times L) \to T^k((1 \times K) \cup (I \times L), (1 \times K) \cup (0 \times L))$$

and defines:

Definition 2.3.4. The connecting homomorphism

$$T^k(L) \to T^{k+1}(K, L)$$

is the negative of the composition

$$T^k(L) \xrightarrow{\kappa^*} T^{k+1}(I \times L, \{0,1\} \times L) \xleftarrow{\cong} T^{k+1}(I \times K, (1 \times K) \cup (0 \times L)) \longrightarrow T^{k+1}(K, L).$$

The last map is induced by the inclusion $(0 \times K, 0 \times L) \to (I \times K, (1 \times K) \cup (0 \times L))$.

Theorem 2.3.5 ([LM09, Theorem (12.11)])**.** *T^* is a cohomology theory.* □

2.4 The Ω-spectrum of an ad theory

In this section we describe the Ω-spectrum associated to an ad theory. A basic definition for spectra is given in part 6.4 of the Appendix. Details of the construction given here can be found in Section 13 of [LM09].

Definition 2.4.1. For $k \geq 0$ let P_k be the semisimplicial set

$$(P_k)_n = \mathrm{ad}^k(\Delta^n)$$

The face maps are induced by that of Δ^n. P_k is equipped with a base point determined by the elements $\emptyset$. Define Q_k to be the geometric realization $|P_k|$.

There is a semisimplicial analog of the Kan suspension, defined as follows:

Definition 2.4.2. Let A be a based semisimplicial set. Then ΣA is the based semisimplicial set with only one 0-simplex $*$, the based set of n simplices is A_{n-1} and the face operators $d_i \colon (\Sigma A)_n \to (\Sigma A)_{n-1}$ agree with those of A for $i < n$ and d_n takes all simplices to $*$.

Lemma 2.4.3 ([LM09, Lemma (13.7)])**.** *There is a natural homeomorphism $\Sigma|A| \cong |\Sigma A|$.* □

Now there is a 1-isomorphism of $\mathbb{Z}$-graded categories

$$\theta \colon \mathcal{C}\mathrm{ell}(\Delta^{n+1}, \partial_{n+1}\Delta^{n+1} \cup \{n+1\}) \to \mathcal{C}\mathrm{ell}(\Delta^n)$$

which is constructed as follows: Let σ be a simplex of Δ^{n+1} which is not in $\partial_{n+1}\Delta^{n+1} \cup \{n+1\}$. Then σ must contain the vertex $n+1$. Now one defines θ to take σ with its canonical orientation to the simplex of Δ^n spanned by all the other vertices of σ with $(-1)^{\dim \sigma - 1}$ times its canonical orientation. The sign ensures that θ is incidence-compatible, so by the reindexing axiom it induces a bijection

$$\theta^* \colon \mathrm{ad}^k(\Delta^n) \to \mathrm{ad}^{k+1}(\Delta^{n+1}, \partial_{n+1}\Delta^{n+1} \cup \{n+1\}).$$

Then the composition

$$\mathrm{ad}^k(\Delta^n) \xrightarrow{\theta^*} \mathrm{ad}^{k+1}(\Delta^{n+1}, \partial_{n+1}\Delta^{n+1} \cup \{n+1\}) \longrightarrow \mathrm{ad}^{k+1}(\Delta^{n+1})$$

defines a semisimplicial map

$$\Sigma P_k \to P_{k+1}.$$

Definition 2.4.4. The spaces Q_k together with the structure maps

$$\Sigma Q_k = \Sigma|P_k| \cong |\Sigma P_k| \to |P_{k+1}| = Q_{k+1}$$

define a spectrum which is denoted by Q.

Proposition 2.4.5 ([LM09, Proposition (13.9)]). *Q is an Ω-spectrum.* □

Remark 2.4.6. The construction of the spectrum associated to an ad theory defines a functor from the category of ad theories to the category of Ω-spectra, which we also will denote by Q.

Theorem 2.4.7 ([LM09, Theorem (14.1)]). *The cohomology theory represented by Q is naturally isomorphic to T^*.* □

2.5 The symmetric spectrum associated to an ad theory

Using multisemisimplicial sets it is possible to construct a symmetric spectrum M associated to an ad theory. A definition for symmetric spectra can be found in part 6.4 of the Appendix. The construction of M is done in section 15 of [LM09].

A k-fold multisemisimplicial set is a contravariant functor from Δ^k_{inj} (see Example 6.5.4) to sets. For a multiindex $\underline{n} = (n_1, \dots, n_k)$, let $\Delta^{\underline{n}}$ be the product $\Delta^{n_1} \times \cdots \times \Delta^{n_k}$. Then the geometric realization of a k-fold multisemisimplicial set A is given by

$$|A| = (\coprod \Delta^{\underline{n}} \times A_{\underline{n}})/\sim$$

with the evident equivalence reltion.

For an ad theory and $k \geq 1$, a k-fold multisemisimplicial set is given by

$$(R_k)_{\underline{n}} = \mathrm{ad}^k(\Delta^{\underline{n}}).$$

Then M_k is defined to be the geometric realization of R_k. For $k = 0$, define R_0 to be the set of $*$-ads of degree 0 and let M_0 be R_0 equipped with the discrete topology.

The action of Σ_k on M_k and the suspension maps are defined in Definition 15.3 and 15.4 of [LM09].

For $k \geq 1$ let Q'_k bet the space homeomorphic to Q_k which is the realization of the semisimplicial set with n-simplices $(R_k)_{(0,\dots,0,n)}$. There are obvious maps $Q'_k \to M_k$, so we get maps

$$Q_k \to M_k.$$

It is possible to show that M is weakly equivalent to Q in the sense of the following two propositions.

Proposition 2.5.1 ([LM09, Proposition (15.7)]). *These maps $Q_k \to M_k$ are weak equivalences.* □

Proposition 2.5.2 ([LM09, Proposition (15.8)]). *The diagrams*

$$\begin{tikzcd} \Sigma Q_k \arrow[r] \arrow[d] & \Sigma M_k \arrow[d] \\ Q_{k+1} \arrow[r] & M_{k+1} \end{tikzcd}$$

commute up to homotopy. □

Let $\mathbb{U}$ be the forgetful functor from symmetric spectra to ordinary spectra. Then $\mathbb{U}$ is a Quillen functor. So its right derived functor $R\mathbb{U}$ exists and it is an equivalence of homotopy categories (see [MMSS01, Lemma A.2]).

Corollary 2.5.3 ([LM09, Corollary (15.9)]).

(i) *M is a positive Ω spectrum, that is, the map $M_k \to \Omega M_{k+1}$ is a weak equivalence for $k \geq 1$.*

(ii) *$R\mathbb{U}$ takes M to Q.*

(iii) *The homotopy groups of M are the bordism groups of the ad theory.* □

2.6 Multiplicative ad theories and symmetric ring spectra

In Section 16 of [LM09] it is shown how to get a symmetric ring spectrum from what is called a multiplicative ad theory. First, a monoidal structure is needed on the target category:

Definition 2.6.1. A *strict monoidal structure* on a $\mathbb{Z}$-graded category $\mathcal{A}$ is a strict monoidal structure $(\boxtimes, \epsilon)$ (as in [ML98, Section VII.1]) on the underlying category which fulfills:

(a) The monoidal product $\boxtimes$ adds dimensions; the dimension of the unit element ϵ is 0.

(b) $i(x \boxtimes y) = (ix) \boxtimes y = x \boxtimes (iy)$ for all objects x and y and similarly for morphisms.

(c) $x \boxtimes \emptyset_n = \emptyset_n \boxtimes x = \emptyset_{n+\dim x}$ for all objects x and all n. If further $f \colon x \to y$ is a morphism then $f \boxtimes \emptyset_n$ and $\emptyset_n \boxtimes f$ are each equal to the canonical map

$$\emptyset_{n+\dim x} \to \emptyset_{n+\dim y}.$$

Remark 2.6.2 ([LM09, Remark (16.3)]). For a $\mathbb{Z}$-graded category $\mathcal{A}$ with a strict monoidal structure, there is a natural map on pre-ads

$$\boxtimes\colon \operatorname{pre}^k(K) \times \operatorname{pre}^l(L) \to \operatorname{pre}^{k+l}(K \times L)$$

given by

$$(F \boxtimes G)(\sigma \times \tau, o_1 \times o_2) = i^{l \dim(\sigma)} F(\sigma, o_1) \boxtimes G(\tau, o_2).$$

This is well-defined, because of property (b) of the definition of a strict monoidal structure.

Definition 2.6.3. An ad theory together with a strict monoidal structure on the target category $\mathcal{A}$ is called a *multiplicative ad theory*, if it fulfills the following two conditions:

(a) The pre $*$-ad with value ϵ is an ad.

(b) The map of the last remark restricts to a map

$$\boxtimes\colon \operatorname{ad}^k(K) \times \operatorname{ad}^l(L) \to \operatorname{ad}^{k+l}(K \times L).$$

Theorem 2.6.4 ([LM09, Theorem (16.5)])**.** *The symmetric spectrum M determined by a multiplicative ad theory is a symmetric ring spectrum.* □

Remark 2.6.5. Note, that a symmetric ring spectrum satisfies strict associativity, not only associativity up to homotopy.

3 The ad theory with values in the category of $\mathcal{C}$-diagrams

It is our aim to construct equivariant ad theories. For manifolds and their bordism theory, groups appear if one generalizes to G-manifolds and G-bordism. Guided by this example, we want to construct ad theories with target categories consisting of G-objects and we want to get G-cohomology theories from this construction, which are linked for different groups by the structure of an equivariant cohomology theory.

As already said in the Introduction, it turns out to be useful to adopt the point of view on equivariant (co)homology theories that is taken in the work of Wolfgang Lück and others ([DL98], [KL05] and [Lüc05]). In particular, it is shown there, how to get G-(co)homology theories from or(G)-spectra and equivariant (co)homology theories from functors from the category of certain small groupoids to the category of (Ω-)spectra, that take equivalences of groupoids to weak equivalences of spectra. The groupoids one is interested in are transport groupoids of homogeneous spaces of the form G/H. We will go into details about all this in Section 4.

For now we keep in mind, that we want to get such functors from groupoids to Ω-spectra. That is, we want to construct new ad theories not only of G-objects, but also of $\mathcal{C}$-objects for certain small categories $\mathcal{C}$. For example $\mathcal{C}$ should be allowed to be such a transport groupoid of some G/H. It will turn out that $\mathcal{C}$ should at least be a small category with exactly one isomorphism class of objects. The transport groupoids of homogeneous spaces of the form G/H satisfy these requirements. In fact they are equivalent to the categories of the groups H.

More precisely we want to take an existing ad theory with target category $\mathcal{A}$ and define a new ad theory with the target category being the functor category $\mathcal{A}^{\mathcal{C}}$ equipped with an appropriate $\mathbb{Z}$-graded structure. This functor category is also called the category of $\mathcal{C}$-diagrams in $\mathcal{A}$; see part 6.5 of the Appendix for basic definitions of diagram categories and evaluation functors.

It seems natural that one wants a K-ad to be a pre-K-ad $\mathcal{C}\mathrm{ell}(K) \to \mathcal{A}^{\mathcal{C}}$ for which the evaluation at every object C of $\mathcal{C}$ is an ad. We will show in this part of the work that for an ad theory fulfilling certain additional properties, this always defines a new ad theory, which is multiplicative if the original ad theory was. The properties mentioned are some kind of functoriality of the gluing and cylinder constructions. They are introduced in Section 3.2. We will see later in Section 5 that all the standard examples for ad theories have these properties.

The example that guides us is given by topological G-manifolds. That is we want to take the ad theory of oriented topological bordism (see Section 5) and get a new ad theory whose target category is the category of G-manifolds (that means $\mathcal{C}$ is the category of the group G). For this example an I-ad defined as above is a G-bordism and so the bordism groups of the new ad theory are the G-bordism groups.

The category $\mathcal{A}_{STop}$ has only inclusions as morphisms, but we need at least homeomorphisms of topological manifolds to describe G-actions. A solution would be to add embeddings to this category (see Definition 2.1.6) and redefine the ad theory, but we take a slightly more general approach, which might be useful for other examples of ad theories, in Section 3.5.

Later we may want to restrict ourselves to the bordism of G-manifolds with restricted isotropy or continuous actions of topological groups. This is also possible by restriction to a subcategory of the diagram category and we provide the techniques necessary for this in Section 3.6.

As mentioned above, throughout this section $\mathcal{C}$ is a small category with exactly one isomorphism class of objects, that is, for each pair of objects in $\mathcal{C}$ there exists an isomorphism between them. We denote by B a fixed representative for the only isomorphism class. All constructions we make will not depend on the choice of B.

The standard example for $\mathcal{C}$ one should think of is the category $\mathcal{G}$ of a group G with its only object $*$. Other examples are given by small categories that are equivalent to $\mathcal{G}$ or by the transport groupoids of homogeneous spaces of the form G/H (in fact such a transport groupoid of G/H is equivalent to the category of the group H, see part 6.3 of the Appendix for foundations of equivariant topology and Section 4.1 for the definition of the transport groupoid of a G-set).

3.1 The $\mathbb{Z}$-grading of the diagram category

The first thing we have to do is to explain how we get a $\mathbb{Z}$-graded structure on $\mathcal{A}^{\mathcal{C}}$. The following Proposition does this:

Proposition 3.1.1. *Let $\mathcal{A}$ be a $\mathbb{Z}$-graded category and $\mathcal{C}$ a small category with exactly one isomorphism class. Let B be a representative for this class. The diagram category $\mathcal{A}^{\mathcal{C}}$ is a $\mathbb{Z}$-graded category with involution*

$$\begin{aligned} i_{\mathcal{A}^{\mathcal{C}}} := i_{\mathcal{A}*} \colon \mathcal{A}^{\mathcal{C}} &\to \mathcal{A}^{\mathcal{C}} \\ F &\mapsto i_{\mathcal{A}} \circ F \\ (\alpha \colon F \to G) &\mapsto i_{\mathcal{A}}(\alpha), \end{aligned}$$

dimension functor

$$d_{\mathcal{A}^{\mathcal{C}}} := d_{\mathcal{A}} \circ \mathrm{ev}_B,$$

and $\emptyset_{\mathcal{A}^{\mathcal{C}}}(n)$ being the constant functor from $\mathcal{C}$ to $\emptyset_{\mathcal{A}}(n)$. Furthermore this construction is functorial in the sense that if $\mathcal{C}' \to \mathcal{C}$ is a functor between small categories with exactly one isomorphism class, then $\mathcal{A}^{\mathcal{C}} \to \mathcal{A}^{\mathcal{C}'}$ is a 0-morphism and if $\mathcal{A}' \to \mathcal{A}$ is a k-morphism then $\mathcal{A}'^{\mathcal{C}} \to \mathcal{A}^{\mathcal{C}}$ is a k-morphism.

Proof. Note that the dimension does not depend on the choice of the representative B, because it is a functor from $\mathcal{A}$ to $\mathbb{Z}$. So the dimension of isomorphic objects in $\mathcal{A}$ agrees. Everything else can be reduced to $\mathcal{A}$ being a $\mathbb{Z}$-graded category. $\square$

We call this $\mathbb{Z}$-graded structure the *induced $\mathbb{Z}$-grading* and from now on we assume that $\mathcal{A}^{\mathcal{C}}$ is equipped with this $\mathbb{Z}$-graded structure.

3.2 Functorial gluing and cylinder constructions

As we intend to construct a new ad theory with target category $\mathcal{A}^{\mathcal{C}}$ from an ad theory with target category $\mathcal{A}$, we need methods to transport the gluing and cylinder constructions, which we have for every evaluation, to diagrams. Therefore we have to require that these constructions are functorial in a certain sense which we want to enlighten in this part.

Definition 3.2.1. For an ad theory with target category $\mathcal{A}$ the set $\mathrm{ad}^k(K)$ can be regarded as a category whose objects are the ads of degree k and whose morphisms are natural transformations between such ads. We call this category the *category of K-ads of degree k* and denote it by $\mathrm{ad}^k(K)$ again. Note that $\mathrm{ad}^k(K)$ is simply the full subcategory of k-ads of the functor category of functors $\mathcal{C}\mathrm{ell}(K) \to \mathcal{A}$.

Definition 3.2.2. An ad theory is called *ad theory with functorial gluing constructions* if for every subdivision K' of K and every $k \in \mathbb{Z}$ there exists a functor $G\colon \mathrm{ad}^k(K') \to \mathrm{ad}^k(K)$ such that G is the identity functor on residual subcomplexes, that is $G(F)$ agrees with F on each residual subcomplex and for a natural transformation $g\colon F_1 \to F_2$ the image $G(g)$ agrees with g on each residual subcomplex.

Now let $\mathcal{C}$ be a small category with exactly one isomorphism class and let $\mathcal{A}$ be the target category of an ad theory $\mathrm{ad}_{\mathcal{A}}$.

Definition 3.2.3. A (K,L)-ad of degree k with values in $\mathcal{A}^{\mathcal{C}}$ is a k-morphism

$$F\colon \mathcal{C}\mathrm{ell}(K,L) \to \mathcal{A}^{\mathcal{C}}$$

such that for each object C of $\mathcal{C}$ the functor $\mathrm{ev}_C \circ F\colon \mathcal{C}\mathrm{ell}(K,L) \to \mathcal{A}$ is a (K,L)-ad. The set of (K,L)-ads of degree k is denoted by $\mathrm{ad}^k_{\mathcal{A}^{\mathcal{C}}}(K,L)$.

Remark 3.2.4. It follows from the definition and the fact that all objects of $\mathcal{C}$ are isomorphic, that such a (K,L)-ad of degree k is an ad of degree k for every evaluation at an object C of $\mathcal{C}$.

Remark 3.2.5. The 0-morphism induced by a functor of small categories with exactly one isomorphism class preserves ads. If $\mathcal{A} \to \mathcal{A}'$ is functor that preserves ads, then the induced functor $\mathcal{A}^{\mathcal{C}} \to \mathcal{A}'^{\mathcal{C}}$ is a functor that preserves ads.

Lemma 3.2.6. *Let K' be a subdivision of a ball complex K and let $\mathcal{A}$ be the target category of an ad theory with functorial gluing constructions. Then for every $F \in \mathrm{ad}^k_{\mathcal{A}^{\mathcal{C}}}(K')$ there is a k-morphism*

$$\widetilde{F}\colon \mathcal{C}\mathrm{ell}(K) \to \mathcal{A}^{\mathcal{C}},$$

given by

$$\widetilde{F}(-)(C) = G(\mathrm{ev}_C \circ F)(-)$$

and for morphisms $g\colon C_1 \to C_2$ by

$$\widetilde{F}(-)(g) = G(\mathrm{ev}_g \circ F)(-),$$

that is $\widetilde{F}(-_1)(-_2) = G(\mathrm{ev}(-_2) \circ F)(-_1)$. This pre-ad is an ad, that is $\widetilde{F} \in \mathrm{ad}^k_{\mathcal{A}^{\mathcal{C}}}(K)$ and $\widetilde{F}$ agrees with F on each residual subcomplex.

Proof. We see in remark 6.5.10 of the appendix that ev defines a functor. Since G is a functor, $G(\mathrm{ev}(-) \circ F)$ defines a functor, so $\widetilde{F}$ is a functor. Now $\widetilde{F}$ is a k-morphism because G takes k-ads to k-ads, so it does that for every evaluation $\mathrm{ev}_C \circ F$ at an object C of $\mathcal{C}$. This also shows that $\mathrm{ev}_C \circ \widetilde{F}$ is an ad for every object C, so $\widetilde{F} \in \mathrm{ad}^k_{\mathcal{A}^{\mathcal{C}}}(K)$ by definition. It is clear that $\widetilde{F}$ agrees with F on residual subcomplexes, because G preserves that. □

Definition 3.2.7. An ad theory is called *ad theory with functorial cylinder constructions* if for every $k \in \mathbb{Z}$ there exists a natural transformation $J : \mathrm{ad}^k(-) \to \mathrm{ad}^k(- \times I)$ which has the following properties:

(i) For each fixed ball complex K it is a functor

$$J(K)\colon\ \mathrm{ad}^k(K) \to \mathrm{ad}^k(K \times I)$$

(ii) For each ball complex K, let

$$\begin{aligned}\mathrm{in}^*_{K\times 1}&\colon\ \mathrm{ad}^k(K \times I) \to \mathrm{ad}^k(K \times 1)\\ \mathrm{in}^*_{K\times 0}&\colon\ \mathrm{ad}^k(K \times I) \to \mathrm{ad}^k(K \times 0)\end{aligned}$$

denote the restriction functors induced by the inclusions of $K \times 1$ and $K \times 0$ to $K \times I$. Then $\mathrm{in}^*_{K\times 1} \circ J(K)$ and $\mathrm{in}^*_{K\times 0} \circ J(K)$ both are the identity functor

$$\mathrm{ad}^k(K) \to \mathrm{ad}^k(K),$$

that is $J(K)(F)$ is equal to F on $K \times 0$ and $K \times 1$ and for a natural transformation $g\colon F_1 \to F_2$ the natural transformation $J(K)(g)$ is equal to g on $K \times 0$ and $K \times 1$.

(iii) J takes trivial ads to trivial ads.

Lemma 3.2.8. *Let $\mathcal{A}$ be the target category of an ad theory with functorial cylinder constructions. Then we can construct a natural transformation*

$$J_{\mathcal{A}^{\mathcal{C}}} : \mathrm{ad}^k_{\mathcal{A}^{\mathcal{C}}}(-) \to \mathrm{ad}^k_{\mathcal{A}^{\mathcal{C}}}(- \times I),$$

which fulfills the cylinder axiom of an ad theory, as follows: For a ball complex K it takes an element $F \in \mathrm{ad}^k_{\mathcal{A}^{\mathcal{C}}}(K)$ to $J_{\mathcal{A}^{\mathcal{C}}}(K)(F)$ which is defined by

$$J_{\mathcal{A}^{\mathcal{C}}}(K)(F)(-)(C) := J_{\mathcal{A}}(K)(\mathrm{ev}_C \circ F)(-)$$

on objects C and

$$J_{\mathcal{A}^{\mathcal{C}}}(K)(F)(-)(g) := J_{\mathcal{A}}(K)(\mathrm{ev}_g \circ F)(-).$$

on morphisms $g\colon C_1 \to C_2$, that is $J_{\mathcal{A}^{\mathcal{C}}}(K)(F)(-_1)(-_2) := J_{\mathcal{A}}(K)(\mathrm{ev}(-_2) \circ F)(-_1)$.

Proof. Completely analogous to the proof of Lemma 3.2.6, $J_{\mathcal{A}^{\mathcal{C}}}(K)(F)$ is a functor. It is clear that $J_{\mathcal{A}^{\mathcal{C}}}$ is a natural transformation of the two functors on the category of ball complexes, because $J_{\mathcal{A}}$ is a natural transformation. Let $\mathrm{in}_{K\times 0} : \mathcal{C}\mathrm{ell}(K\times 0) \to \mathcal{C}\mathrm{ell}(K\times I)$ be the inclusion functor. Then we can use that $J_{\mathcal{A}}$ is a cylinder construction and get $J_{\mathcal{A}^{\mathcal{C}}}(K)(F)(\mathrm{in}_{K\times 0}(-))(C) = J_{\mathcal{A}}(K)(\mathrm{ev}_C \circ F)(\mathrm{in}_{K\times 0}(-)) = \mathrm{ev}_C \circ F = F(-)(C)$. Because $J_{\mathcal{A}}$ is a functorial cylinder construction we even get $J_{\mathcal{A}^{\mathcal{C}}}(K)(F)(\mathrm{in}_{K\times 0}(-))(g) = J_{\mathcal{A}}(K)(\mathrm{ev}_g \circ F)(\mathrm{in}_{K\times 0}(-)) = \mathrm{ev}_g \circ F$ for morphisms $g\colon C_1 \to C_2$. Hence the functor $J_{\mathcal{A}^{\mathcal{C}}}(K)(F)$ restricts to the functor F on $K \times 0$. The same proof can be applied to the case $K \times 1$. Trivial ads are taken to trivial ads by construction. □

3.3 The ad theory with values in $\mathcal{C}$-diagrams

Now we are ready to prove the main theorem of this section:

Theorem 3.3.1. *Let $\mathcal{A}$ be the target category of an ad theory with functorial gluing and cylinder constructions and let $\mathcal{C}$ be a small category with exactly one isomorphism class. Let $\mathcal{A}^{\mathcal{C}}$ denote the functor category with the $\mathbb{Z}$-graded structure of Proposition 3.1.1. Then the sets*

$$ad^k_{\mathcal{A}^{\mathcal{C}}}(K, L)$$

define subfunctors of $\mathrm{pre}^k_{\mathcal{A}^{\mathcal{C}}}$ *which define an ad theory with target category $\mathcal{A}^{\mathcal{C}}$. Furthermore this construction is a functor on the full subcategory of ad theories with functorial gluing and cylinder constructions to the category of ad theories.*

Proof. The axioms and even the fact that the sets define a subfunctor are checked by reducing everything to the consideration of evaluations at objects C of $\mathcal{C}$. Difficulties only arise when one has to check the axioms which construct new ads: These are the gluing and the cylinder axiom. Here we need the functoriality of these constructions. For gluing we can apply Lemma 3.2.6 and for the cylinder axiom we simply have to apply Lemma 3.2.8. By Remark 3.2.5 this defines a functor from ad theories with functorial gluing and cylinder constructions to the category of ad theories. □

We call this new ad theory the *associated ad theory of $\mathcal{C}$-diagrams in $\mathcal{A}$*.

3.4 Multiplicativity of the ad theory with values in $\mathcal{C}$-diagrams

For a multiplicative ad theory with functorial gluing and cylinder constructions and target category $\mathcal{A}$ it is possible to equip the associated ad theory of $\mathcal{C}$-diagrams with a multiplicative structure, too.

First we give the diagram category $\mathcal{A}^{\mathcal{C}}$ the induced strict monoidal structure. See 2.6.1 for the definition of strict monoidal structures on $\mathbb{Z}$-graded categories.

Lemma 3.4.1. *Let $\mathcal{C}$ be a small category with exactly one isomorphism class and let $\mathcal{A}$ be a $\mathbb{Z}$-graded category with strict monoidal structure $(\boxtimes, \epsilon)$. Let $\mathcal{A}^{\mathcal{C}}$ be equipped with the* *induced $\mathbb{Z}$-graded structure of proposition 3.1.1. Then $\mathcal{A}^{\mathcal{C}}$ has a strict monoidal structure given by*

$$\boxtimes_{\mathcal{C}} \colon \mathcal{A}^{\mathcal{C}} \times \mathcal{A}^{\mathcal{C}} \to \mathcal{A}^{\mathcal{C}}$$
$$(F, G) \mapsto \begin{cases} (C \mapsto F(C) \boxtimes G(C)) \\ (g \colon C_1 \to C_2) \mapsto F(g) \boxtimes G(g) \end{cases}$$

and $\epsilon_{\mathcal{C}}$ is the constant functor to ϵ.

Proof. It is clear that $\boxtimes_{\mathcal{C}}$ is a bifunctor because $\boxtimes$ is one. It is strictly associative because the functors $(-) \boxtimes_{\mathcal{A}^{\mathcal{C}}} ((-) \boxtimes_{\mathcal{A}^{\mathcal{C}}} (-))$ and $((-) \boxtimes_{\mathcal{A}^{\mathcal{C}}} (-)) \boxtimes_{\mathcal{A}^{\mathcal{C}}} (-)$ are equal on objects C of $\mathcal{C}$ and on morphisms $g \colon C_1 \to C_2$ and they are equal on natural transformations of diagrams all because $\boxtimes$ is strictly associative. By definition of $\boxtimes_{\mathcal{C}}$ the diagram $\epsilon_{\mathcal{C}}$ is the unit object and left and right identity are strict because they are for ϵ and $\boxtimes$.

The dimension of a diagram was defined to be the dimension of the evaluation at one of the objects of $\mathcal{C}$ (Recall that it does not depend on the choice of the object because there is only one isomorphism class of objects in $\mathcal{C}$). Thus it is clear that $\boxtimes_{\mathcal{C}}$ adds dimensions because $\boxtimes$ does and the dimension of the unit element $\epsilon_{\mathcal{C}}$ is that of ϵ, so it is 0. The involution on diagrams was defined to be the composition with the involution i of $\mathcal{A}$. So we can use that $(\boxtimes, \epsilon)$ is a strict monoidal structure to show that

$$i_{\mathcal{A}^{\mathcal{C}}}(F \boxtimes_{\mathcal{C}} G) = (i_{\mathcal{A}^{\mathcal{C}}} F) \boxtimes_{\mathcal{C}} G = F \boxtimes_{\mathcal{C}} (i_{\mathcal{A}^{\mathcal{C}}} G)$$

and for natural transformations f, g of diagrams

$$i_{\mathcal{A}^{\mathcal{C}}}(f \boxtimes_{\mathcal{C}} g) = (i_{\mathcal{A}^{\mathcal{C}}} f) \boxtimes_{\mathcal{C}} g = f \boxtimes_{\mathcal{C}} (i_{\mathcal{A}^{\mathcal{C}}} g).$$

Similarly one uses that $\emptyset_{\mathcal{A}^{\mathcal{C}}}(n)$ is the constant diagram of the $\emptyset_{\mathcal{A}}(n)$ to show that

$$F \boxtimes_{\mathcal{C}} \emptyset_{\mathcal{A}^{\mathcal{C}}} = \emptyset_{\mathcal{A}^{\mathcal{C}}} \boxtimes_{\mathcal{C}} F = \emptyset_{\mathcal{A}^{\mathcal{C}}}(n + \dim(F)).$$

and that for natural transformations of diagrams $f \colon F_1 \to F_2$ the natural transformations $f \boxtimes_{\mathcal{C}} \emptyset_{\mathcal{A}^{\mathcal{C}}}(n)$ and $\emptyset_{\mathcal{A}^{\mathcal{C}}}(n) \boxtimes_{\mathcal{C}} f$ are both equal to the canonical morphism $\emptyset_{\mathcal{A}^{\mathcal{C}}}(n + \dim(F_1)) \to \emptyset_{\mathcal{A}^{\mathcal{C}}}(n + \dim(F_2))$. □

We call this strict monoidal structure the *induced strict monoidal structure.*

Theorem 3.4.2. *Let $\mathcal{A}$ be the target category of a multiplicative ad theory with functorial gluing and cylinder constructions. Then the associated ad theory of $\mathcal{C}$-diagrams in $\mathcal{A}$ is again a multiplicative ad theory.*

Proof. Let E be the pre $*$-ad with value $\epsilon_{\mathcal{C}}$. Then by definition every evaluation at an object of $\mathcal{C}$ is the pre $*$-ad with value ϵ which is an ad because the ad theory over $\mathcal{A}$ was multiplicative. Hence E is an ad.

We also have to show that the natural map

$$\mathrm{pre}^{k}_{\mathcal{A}^{\mathcal{C}}}(K) \times \mathrm{pre}^{l}_{\mathcal{A}^{\mathcal{C}}}(L) \to \mathrm{pre}^{k+l}_{\mathcal{A}^{\mathcal{C}}}(K \times L)$$

of remark 2.6.2 restricts to a map

$$\mathrm{ad}^k_{\mathcal{A}^\mathcal{C}}(K) \times \mathrm{ad}^l_{\mathcal{A}^\mathcal{C}}(L) \to \mathrm{ad}^{k+l}_{\mathcal{A}^\mathcal{C}}(K \times L).$$

For that we only have to show that the image of a pair of ads over $\mathcal{A}^\mathcal{C}$ is an ad with values in $\mathcal{A}$ for every evaluation. This is true because of the definition of $\boxtimes_\mathcal{C}$ and the fact, that the ad theory with values in $\mathcal{A}$ was multiplicative. ☐

3.5 Generalization to more morphisms

The example that guides us are G-manifolds. The category $\mathcal{A}_{STop}$ has the disadvantage that all the morphisms are inclusions, but we need at least (orientation-preserving) homeomorphisms to describe G-actions on manifolds. That is we have to add for example embeddings to $\mathcal{A}_{STop}$. Of course it would be possible to redefine the ad theory with this new target category, but that is not necessary: Here we want to solve this problem in a more general manner and hope that this may be helpful for other examples in the future.

Basically we allow the $\mathcal{C}$-diagrams to be functors from $\mathcal{C}$ to a $\mathbb{Z}$-graded category with more morphisms than $\mathcal{A}$, while the morphisms between such $\mathcal{C}$-diagrams are restricted to those natural transformations whose morphisms are morphisms of $\mathcal{A}$. Then we adapt what we have proven before to this new situation.

Let $\mathcal{A}$ be a $\mathbb{Z}$-graded category which is a $\mathbb{Z}$-graded subcategory of a $\mathbb{Z}$-graded category $\mathcal{D}$ and contains all objects of $\mathcal{D}$. Let $\mathcal{D}^\mathcal{C}$ be the diagram category for a small category $\mathcal{C}$ with exactly one isomorphism class of objects. Let B be an arbitrary representative for this isomorphism class.

Definition 3.5.1. We denote by $\mathcal{D}^\mathcal{C}_\mathcal{A}$ the subcategory of $\mathcal{D}^\mathcal{C}$ which consists of all objects of $\mathcal{D}^\mathcal{C}$ and whose morphisms are the natural transformations $\phi\colon F \to G$ for which the morphisms $\phi_C\colon F(C) \to G(C)$ are morphisms of $\mathcal{A}$.

Proposition 3.5.2. *The category $\mathcal{D}^\mathcal{C}_\mathcal{A}$ is a $\mathbb{Z}$-graded category with involution given by the restriction of*

$$\begin{aligned} i_{\mathcal{D}^\mathcal{C}} := i_{\mathcal{D}*}\colon \mathcal{D}^\mathcal{C} &\to \mathcal{D}^\mathcal{C} \\ F &\mapsto i_\mathcal{D} \circ F \\ (\alpha\colon F \to G) &\mapsto i_\mathcal{D}(\alpha), \end{aligned}$$

to $\mathcal{D}^\mathcal{C}_\mathcal{A}$ and dimension functor

$$d_{\mathcal{D}^\mathcal{C}_\mathcal{A}} := d_\mathcal{A} \circ \mathrm{ev}_B,$$

and $\emptyset_{\mathcal{D}^\mathcal{C}_\mathcal{A}}(n)$ being the constant functor from $\mathcal{C}$ to $\emptyset_\mathcal{A}(n)$.

Proof. The proof is essentially the same as that for 3.1.1. Note again, that the dimension does not depend on the choice of B. We can apply $d_\mathcal{A}$ to the evaluation ev_B because of the restriction to natural transformations we made. It ensures, that after evaluating the morphisms are morphisms of $\mathcal{A}$. Thus $d_{\mathcal{D}^\mathcal{C}_\mathcal{A}}$ defines a functor. ☐

Now let $\mathcal{A}$ additionally be the target category of an ad theory.

Definition 3.5.3. Every (K, L)-ad of degree k with values in $\mathcal{A}$ is a functor $\mathcal{C}\mathrm{ell}(K) \to \mathcal{A}$. If we compose it with the inclusion of categories $j\colon \mathcal{A} \to \mathcal{D}$ we get a functor $\mathcal{C}\mathrm{ell}(K) \to \mathcal{D}$. Therefore we can define a category whose objects are the (K, L)-ads of degree k and whose morphisms are the natural transformations of the compositions of such ads with the inclusion functor to $\mathcal{D}$. We call this category the *category of (K, L)-ads of degree k with respect to $\mathcal{D}$* and denote it by $\mathrm{ad}^{k,\mathcal{D}}(K, L)$.

Definition 3.5.4. The ad theory with target category $\mathcal{A}$ is called *ad theory with functorial gluing construction with respect to $\mathcal{D}$* if for each subdivision K' of K and every $k \in \mathbb{Z}$ there exists a functor $G\colon \mathrm{ad}^{k,\mathcal{D}}(K') \to \mathrm{ad}^{k,\mathcal{D}}(K)$ such that G is the identity functor on residual subcomplexes, that is $G(F)$ agrees with F on each residual subcomplex and for a natural transformation $g\colon j \circ F_1 \to j \circ F_2$ the image $G(g)$ agrees with g on each residual subcomplex.

Definition 3.5.5. A (K, L)-ad of degree k with values in $\mathcal{D}_{\mathcal{A}}^{\mathcal{C}}$ is a k-morphism

$$F\colon \mathcal{C}\mathrm{ell}(K, L) \to \mathcal{D}_{\mathcal{A}}^{\mathcal{C}}$$

such that for each object C of $\mathcal{C}$ the functor $\mathrm{ev}_C \circ F\colon \mathcal{C}\mathrm{ell}(K, L) \to \mathcal{A}$ is a (K, L)-ad of degree k. The set of (K, L)-ads of degree k is denoted by $\mathrm{ad}^k_{\mathcal{D}_{\mathcal{A}}^{\mathcal{C}}}(K, L)$.

Remark 3.5.6. A functor $\mathcal{C} \to \mathcal{C}'$ between small categories with exactly one isomorphism class induces a morphism of $\mathbb{Z}$-graded categories $\mathcal{D}_{\mathcal{A}}^{\mathcal{C}} \to \mathcal{D}_{\mathcal{A}}^{\mathcal{C}'}$ which preserves ads.

Example 3.5.7. A natural transformation between two K-ads F and G in $\mathrm{ad}^k_{\mathcal{D}_{\mathcal{A}}^{\mathcal{C}}}(K)$ is given by morphisms $F(\sigma, o)(C) \to G(\sigma, o)(C)$ in $\mathcal{A}$ such that for all morphisms $C_1 \to C_2$ in $\mathcal{C}$ and $(\tau, o') \to (\sigma, o)$ in $\mathcal{C}\mathrm{ell}(K)$ the diagram

$$\begin{tikzcd}
& F(\sigma,o)(C_1) \arrow[rr] \arrow[dd] & & F(\sigma,o)(C_2) \arrow[dd] \\
F(\tau,o')(C_1) \arrow[ur] \arrow[rr] \arrow[dd] & & F(\tau,o')(C_2) \arrow[ur] \arrow[dd] & \\
& G(\sigma,o)(C_1) \arrow[rr] & & G(\sigma,o)(C_2) \\
G(\tau,o')(C_1) \arrow[ur] \arrow[rr] & & G(\tau,o')(C_2) \arrow[ur] &
\end{tikzcd}$$

commutes. The horizontal arrows are morphisms in $\mathcal{D}$ and all other arrows have to be morphisms in $\mathcal{A}$. It is helpful to have this diagram in mind to recognize where the morphisms have to be in $\mathcal{D}$ and where they have to be in $\mathcal{A}$.

Lemma 3.5.8. *Let K' be a subdivision of a ball complex K and let $\mathcal{A}$ be the target category of an ad theory with functorial gluing construction with respect to $\mathcal{D}$. Then for every $F \in \mathrm{ad}^k_{\mathcal{D}_{\mathcal{A}}^{\mathcal{C}}}(K')$ there is a k-morphism*

$$\widetilde{F}\colon \mathcal{C}\mathrm{ell}(K) \to \mathcal{D}_{\mathcal{A}}^{\mathcal{C}},$$

given by

$$\widetilde{F}(-)(C) = G(\mathrm{ev}_C \circ F)(-)$$

and for morphisms $g\colon C_1 \to C_2$ *by*

$$\widetilde{F}(-)(g) = G(\mathrm{ev}_g \circ F)(-),$$

that is $\widetilde{F}(-_1)(-_2) = G(\mathrm{ev}(-_2) \circ F)(-_1)$. *This pre-ad is an ad, that is* $\widetilde{F} \in \mathrm{ad}^k_{\mathcal{D}^{\mathcal{C}}_{\mathcal{A}}}(K)$ *and* $\widetilde{F}$ *agrees with* F *on each residual subcomplex.*

Proof. The proof is completely analogous to the proof of 3.2.6, we only have to mention that $\mathrm{ev}_g \circ F$ is a natural transformation of $j \circ \mathrm{ev}_{C_1} \circ F$ to $j \circ \mathrm{ev}_{C_2} \circ F$ and therefore we can apply G to it. $\square$

Definition 3.5.9. An ad theory with target category $\mathcal{A}$ is called *ad theory with functorial cylinder construction with respect to* $\mathcal{D}$ if for every $k \in \mathbb{Z}$ there exists a natural transformation $J : \mathrm{ad}^k(-) \to \mathrm{ad}^k(- \times I)$ which has the following properties:

(i) For each ball complex K it is a functor

$$J(K)\colon\ \mathrm{ad}^{k,\mathcal{D}}(K) \to \mathrm{ad}^{k,\mathcal{D}}(K \times I)$$

(ii) For each ball complex K the restrictions of this functor $J(K)$ to $K \times 0$ and $K \times 1$ are the identity, that is $J(K)(F)$ is F on $K \times 0$ and $K \times 1$ and for a natural transformation $g\colon j \circ F_1 \to j \circ F_2$ the natural transformation $J(K)(g)$ is equal to g on $K \times 0$ and $K \times 1$.

(iii) J takes trivial ads to trivial ads.

Lemma 3.5.10. *Let* $\mathcal{A}$ *be the target category of an ad theory with functorial cylinder construction with respect to* $\mathcal{D}$. *Then we can construct a natural transformation*

$$J_{\mathcal{D}^{\mathcal{C}}_{\mathcal{A}}} : \mathrm{ad}^k_{\mathcal{D}^{\mathcal{C}}_{\mathcal{A}}}(-) \to \mathrm{ad}^k_{\mathcal{D}^{\mathcal{C}}_{\mathcal{A}}}(- \times I),$$

which fulfills the cylinder axiom of an ad theory, as follows: For a ball complex K *it takes an element* $F \in \mathrm{ad}^k_{\mathcal{D}^{\mathcal{C}}_{\mathcal{A}}}(K)$ *to* $J_{\mathcal{D}^{\mathcal{C}}_{\mathcal{A}}}(K)(F)$ *which is defined by*

$$J_{\mathcal{D}^{\mathcal{C}}_{\mathcal{A}}}(K)(F)(-)(C) := J_{\mathcal{A}}(K)(\mathrm{ev}_C \circ F)(-)$$

on objects C *and*

$$J_{\mathcal{D}^{\mathcal{C}}_{\mathcal{A}}}(K)(F)(-)(g) := J_{\mathcal{A}}(K)(\mathrm{ev}_g \circ F)(-).$$

on morphisms $g\colon C_1 \to C_2$, *that is* $J_{\mathcal{D}^{\mathcal{C}}_{\mathcal{A}}}(K)(F)(-_1)(-_2) := J_{\mathcal{A}}(K)(\mathrm{ev}(-_2) \circ F)(-_1)$.

Proof. Again we only have to mention that $J_{\mathcal{A}}(K)$ can be applied to $\mathrm{ev}_C \circ F$ and to $\mathrm{ev}_g \circ F$ because it is a functor on $\mathrm{ad}^{k,\mathcal{D}}(K)$. The rest of the proof is analogous to the proof of 3.2.8. $\square$

Theorem 3.5.11. *Let $\mathcal{A}$ be the target category of an ad theory with functorial gluing and cylinder constructions with respect to $\mathcal{D}$. Then the sets*

$$\mathrm{ad}^k_{\mathcal{D}^{\mathcal{C}}_{\mathcal{A}}}(K,L)$$

define subfunctors of $\mathrm{pre}^k_{\mathcal{D}^{\mathcal{C}}_{\mathcal{A}}}$ which define an ad theory with target category $\mathcal{D}^{\mathcal{C}}_{\mathcal{A}}$.

Proof. This proof is completely analogous to the proof of 3.3.1. □

We call this ad theory the *associated ad theory with values in $\mathcal{D}^{\mathcal{C}}_{\mathcal{A}}$*.

Now we assume that $\mathcal{D}$ has a strict monoidal structure, which restricts to a strict monoidal structure $(\boxtimes, \epsilon)$ of the $\mathbb{Z}$-graded subcategory $\mathcal{A}$.

Lemma 3.5.12. *The $\mathbb{Z}$-graded category $\mathcal{D}^{\mathcal{C}}_{\mathcal{A}}$ has a strict monoidal structure given by*

$$\boxtimes_{\mathcal{D}^{\mathcal{C}}} \colon \mathcal{D}^{\mathcal{C}}_{\mathcal{A}} \times \mathcal{D}^{\mathcal{C}}_{\mathcal{A}} \to \mathcal{D}^{\mathcal{C}}_{\mathcal{A}}$$
$$(F,G) \mapsto \begin{cases} (C \mapsto F(C) \boxtimes G(C)) \\ (g \colon C_1 \to C_2) \mapsto F(g) \boxtimes G(g) \end{cases}$$

and $\epsilon_{\mathcal{D}^{\mathcal{C}}}$ is the constant functor to ϵ.

Proof. The fact that the monoidal product functor of $\mathcal{D}$ restricts to the one of $\mathcal{A}$ ensures that $\boxtimes_{\mathcal{D}^{\mathcal{C}}}$ is a well-defined bifunctor. Strict associativity, left and right identity, and the compatibility with dimension, involution, and with $\emptyset$ follow as in the proof of 3.4.1. □

We call this strict monoidal structure the *induced strict monoidal structure* on $\mathcal{D}^{\mathcal{C}}_{\mathcal{A}}$.

Theorem 3.5.13. *If $\mathcal{A}$ is the target category of a multiplicative ad theory with functorial gluing and cylinder constructions with respect to $\mathcal{D}$, then the ad theory with target category $\mathcal{D}^{\mathcal{C}}_{\mathcal{A}}$ equipped with the induced strict monoidal structure is again a multiplicative ad theory.*

Proof. Completely analogous to 3.4.2. □

3.6 The ad theory associated to a full subcategory of $\mathcal{D}^{\mathcal{C}}_{\mathcal{A}}$

For a topological group G one is interested in the continuous actions on oriented topological manifolds. A continuous action can be described as a continuous functor from the topological category of the group $\mathcal{G}$ to the topological category of oriented topological manifolds. Hence in the context of ad theories we are interested in ad theories with values in the full subcategory of the diagram category which consists of the continuous functors.

Another case where we want a restriction to a full subcategory is given by the bordism of G-manifolds with restricted isotropy.

In this section we reformulate the results of the last section and construct an associated ad theory for such restrictions.

As in the section before, let $\mathcal{A}$ be a $\mathbb{Z}$-graded category which is a $\mathbb{Z}$-graded subcategory of a $\mathbb{Z}$-graded category $\mathcal{D}$ and contains all objects of $\mathcal{D}$. Let $\mathcal{D}^{\mathcal{C}}$ be the diagram category for a small category $\mathcal{C}$ with exactly one isomorphism class of objects and Let $\mathcal{D}^{\mathcal{C}}_{\mathcal{A}}$ be the category of Definition 3.5.1.

Now let $\mathcal{W}$ be a full subcategory of $\mathcal{D}^{\mathcal{C}}_{\mathcal{A}}$ such that the involution and the functor $\emptyset$ restrict to this subcategory. Then it is clear that $\mathcal{W}$ is a $\mathbb{Z}$-graded category. We define the ads with values in $\mathcal{W}$ as expected:

Definition 3.6.1. A (K, L)-ad of degree k with values in $\mathcal{W}$ is a pre-(K, L)-ad of degree k such that after composition with the inclusion functor $\mathcal{W} \to \mathcal{D}^{\mathcal{C}}_{\mathcal{A}}$ it is an ad with values in $\mathcal{D}^{\mathcal{C}}_{\mathcal{A}}$. Hence all the evaluations at objects of $\mathcal{C}$ have to be ads. We denote the set of ads of degree k with values in $\mathcal{W}$ by $\mathrm{ad}^k_{\mathcal{W}}(K, L)$.

Theorem 3.6.2. *Let $\mathcal{A}$ be the target category of an ad theory with functorial gluing and cylinder constructions with respect to $\mathcal{D}$. Then the sets*

$$\mathrm{ad}^k_{\mathcal{W}}(K, L)$$

define subfunctors of $\mathrm{pre}^k_{\mathcal{W}}$ and if both the gluing construction given by Lemma 3.5.8 and the cylinder construction given by Lemma 3.5.10 restrict to ads with values in $\mathcal{W}$, then this defines an ad theory with target category $\mathcal{W}$.

Proof. The assumption that the gluing and cylinder constructions take ads with values in $\mathcal{W}$ to ads with values in $\mathcal{W}$ ensures, that we can simply adopt the proof of 3.5.11. $\square$

Next we investigate multiplicativity. Let $\mathcal{A}$ be the target category of a multiplicative ad theory with functorial gluing and cylinder constructions with respect to $\mathcal{D}$. As in the last section we assume that $\mathcal{A}$ has a strict monoidal structure $(\boxtimes, \epsilon)$ which is the restriction of a strict monoidal structure of the $\mathbb{Z}$-graded category $\mathcal{D}$ to $\mathcal{A}$.

Theorem 3.6.3. *If the induced strict monoidal structure on $\mathcal{D}^{\mathcal{C}}_{\mathcal{A}}$ of Lemma 3.5.12 restricts to a strict monoidal structure of $\mathcal{W}$, that is the bifunctor $\boxtimes_{\mathcal{D}^{\mathcal{C}}}$ restricts to a bifunctor $\boxtimes_{\mathcal{W}} \colon \mathcal{W} \times \mathcal{W} \to \mathcal{W}$ and the constant functor from $\mathcal{C}$ to ϵ is an object of $\mathcal{W}$, then the ad theory with values in $\mathcal{W}$ is multiplicative.*

Proof. The proof is completely analogous to the proof of 3.5.13. $\square$

4 Equivariant theories and $\mathrm{or}(G)$-spectra associated to an ad theory

Here we explain how to get an equivariant cohomology theory from an ad theory with functorial gluing and cylinder constructions. First we introduce equivariant homology and cohomology theories in the sense of [DL98], [KL05] and [Lüc05] and explain how they arise from a functor from the category of certain small groupoids to (Ω-)spectra, which takes equivalences to weak equivalences. Then we apply this to ad theories with functorial gluing and cylinder constructions.

We will use basic concepts of equivariant topology like G-spaces, G-CW-complexes, fixed point spaces, the orbit category $\mathrm{or}(G)$ of a group G and the induction functor. We give an introduction to these concepts in part 6.3 of the Appendix. Detailed explanations can be found in the books [tD87] and [May96].

4.1 Equivariant (co)homology theories and $\mathrm{or}(G)$-spectra

Here we introduce equivariant homology and cohomology theories in the sense of [KL05, ch. 20], [Lüc05], or [DL98] respectively. Let G be a group and Λ a commutative ring with unit.

Definition 4.1.1. A *G-homology theory H_*^G with values in Λ-modules* is a collection of covariant functors H_n^G, $n \in \mathbb{Z}$ from the category of G-CW-pairs to the category of Λ-modules together with natural transformations

$$\partial_n^G(X,A)\colon H_n^G(X,A) \to H_{n-1}^G(A) := H_{n-1}^G(A,\emptyset)$$

such that the following axioms are fulfilled:

(i) (G-homotopy invariance) If two G-maps $f, f'\colon (X,A) \to (Y,B)$ of G-CW-pairs are G-homotopic, then $H_n^G(f) = H_n^G(f')$ for all $n \in Z$.

(ii) (Exactness) For every pair (X,A) of G-CW-complexes, there is a long exact sequence

$$\dots \xrightarrow{H_{n+1}^G(j)} H_{n+1}^G(X,A) \xrightarrow{\partial_{n+1}^G} H_n^G(A) \xrightarrow{H_n^G(i)} H_n^G(X)$$

$$\xrightarrow{H_n^G(j)} H_n^G(X,A) \xrightarrow{\partial_n^G} H_{n-1}^G(A) \xrightarrow{H_{n-1}^G(i)} \dots$$

where $i\colon A \to X$ and $j\colon X \to (X,A)$ are the inclusions.

(iii) (Excision) For a G-CW pair (X,A) and a cellular G-map $f\colon A \to B$ of G-CW-complexes, let $(X \cup_f B, B)$ be equipped with the induced structure of a G-CW-pair. Then the canonical map $(X,A) \to (X \cup_f B, B)$ induces for each $n \in \mathbb{Z}$ an isomorphism

$$H_n^G(X,A) \xrightarrow{\cong} H_n^G(X \cup_f B, B).$$

(iv) (Disjoint union axiom) For a family $X_{i,i\in I}$ of G-CW-complexes, the map induced by the canonical inclusions $j_i\colon X_i \to \coprod_{i\in I} X_i$ is a bijection

$$\bigoplus_{i\in I} H_n^G(j_i)\colon \bigoplus_{i\in I} H_n^G(X_i) \xrightarrow{\cong} H_n^G(\coprod_{i\in I} X_i)$$

for all $n \in \mathbb{Z}$.

Definition 4.1.2. A *G-cohomology theory H_*^G with values in Λ-modules* is a collection of contravariant functors H_G^n, $n \in \mathbb{Z}$ from the category of G-CW-pairs to the category of Λ-modules together with natural transformations

$$\delta_G^n\colon H_G^n(A) \to H_G^{n+1}(X, A)$$

such that the following axioms are fulfilled:

(i) (G-homotopy invariance) If two G-maps $f, f'\colon (X, A) \to (Y, B)$ of G-CW-pairs are G-homotopic, then $H_G^n(f) = H_G^n(f')$ for all $n \in Z$.

(ii) (Exactness) For every pair (X, A) of G-CW-complexes, there is a long exact sequence

$$\dots \xrightarrow{\delta_G^{n-1}} H_G^n(X, A) \xrightarrow{H_G^n(j)} H_G^n(X) \xrightarrow{H_G^n(i)} H_G^n(A)$$

$$\xrightarrow{\delta_G^n} H_G^{n+1}(X, A) \xrightarrow{H_G^{n+1}(j)} H_G^{n+1}(X) \xrightarrow{H_G^{n+1}(i)} \dots$$

where $i\colon A \to X$ and $j\colon X \to (X, A)$ are the inclusions.

(iii) (Excision) For a G-CW pair (X, A) and a cellular G-map $f\colon A \to B$ of G-CW-complexes, let $(X \cup_f B, B)$ be equipped with the induced structure of a G-CW-pair. Then the canonical map $(X, A) \to (X \cup_f B, B)$ induces for each $n \in \mathbb{Z}$ an isomorphism

$$H_G^n(X \cup_f B, B) \xrightarrow{\cong} H_G^n(X, A).$$

(iv) (Disjoint union axiom) For a family $X_{i,i\in I}$ of G-CW-complexes, the map induced by the canonical inclusions $j_i\colon X_i \to \coprod_{i\in I} X_i$ is a bijection

$$\prod_{i\in I} H_G^n(j_i)\colon H_G^n(\coprod_{i\in I} X_i) \xrightarrow{\cong} \prod_{i\in I} H_G^n(X_i)$$

for all $n \in \mathbb{Z}$.

Note, that if G is the trivial group, both definitions agree with the definition of a (co)homology theory (which fulfills the disjoint union axiom) in the non-equivariant sense. The disjoint union axiom is demanded here for the following lemmas to be true.

Lemma 4.1.3 ([KL05, Lemma 20.5]). *Let H_*^G be a G-homology theory. Then for every G-CW-complex X and a directed system $\{X_i \mid i \in I\}$ of G-CW-subcomplexes directed by inclusion such that $X = \bigcup_{i \in I} X_i$, the natural map*

$$\operatorname{colim}_{i \in I} H_n^G(X_i) \xrightarrow{\cong} H_n^G(X)$$

is bijective for all $n \in \mathbb{Z}$. □

Lemma 4.1.4 ([Lüc05, Lemma 1.1]).

(i) Let H_G^* be a G-cohomology theory. Then for every G-CW-pair (X, A) with an exhaustion by subcomplexes $A = X_{-1} \subseteq X_1 \subseteq \ldots \subseteq \bigcup_{n \geq -1} X_n = X$ there is a natural short exact sequence

$$0 \to \lim_{n \to \infty}{}^1 H_G^{p-1}(X_n \cup A, A) \to H^p(X, A) \to \lim_{n \to \infty} H_G^p(X_n \cup A, A) \to 0.$$

(ii) Let H_G^* and K_G^* be G-cohomology theories and $T^*\colon H_G^* \to K_G^*$ a natural transformation of cohomology theories, that is for every $n \in \mathbb{Z}$ we have a natural transformation $T^n\colon H_G^n \to K_G^n$ and these are compatible with the boundary operator. Then if $T^n(G/H)$ is bijective for every homogeneous space G/H and every $n \in \mathbb{Z}$, then $T^n(X, A)\colon H_G^n(X, A) \to K_G^n(X, A)$ is bijective for all $n \in Z$. □

For the next definitions recall Definition 6.3.9 of the induction of a group homomorphism.

Definition 4.1.5. An *equivariant homology theory $H_*^?$ with values in Λ-modules* consists of a G-homology theory H_*^G for every group G together with a so-called *induction structure*: For every group homomorphism $\alpha\colon H \to G$ and H-CW-pair (X, A) such that $\ker(\alpha)$ acts freely on X, there exists a natural isomorphism

$$\operatorname{ind}_\alpha\colon H_n^H(X, A) \xrightarrow{\cong} H_n^G(\operatorname{ind}_\alpha(X, A))$$

for every $n \in \mathbb{Z}$, such that the following conditions are fulfilled:

(i) (Compatibility with the boundary homomorphism) $\partial_n^G \circ \operatorname{ind}_\alpha = \operatorname{ind}_\alpha \circ \partial_n^H$ for all $n \in \mathbb{Z}$.

(ii) (Functoriality) If $\beta\colon G \to K$ is another group homomorphism such that $\ker(\beta \circ \alpha)$ acts freely on X, then for all $n \in \mathbb{Z}$ we have

$$\operatorname{ind}_{\beta \circ \alpha} = H_n^K(f_1) \circ \operatorname{ind}_\beta \circ \operatorname{ind}_\alpha$$

where $f_1\colon \operatorname{ind}_\beta(\operatorname{ind}_\alpha(X, A)) \to \operatorname{ind}_{\beta \circ \alpha}(X, A)$ is the natural K-homeomorphism given by $(k, g, x) \mapsto (k\beta(g), x)$.

(iii) (Compatibility with conjugation) Let $g \in G$ and let $c(g)\colon G \to G$ be the conjugation homomorphism $g' \mapsto gg'g^{-1}$. If (X, A) is a G-CW-pair, then for all $n \in \mathbb{Z}$, the homomorphism

$$\mathrm{ind}_{c(g)\colon G\to G}\colon H_n^G(X, A) \to H_n^G(\mathrm{ind}_{c(g)\colon G\to G}(X, A))$$

agrees with $H_n^G(f_2)$ where f_2 is the G-homeomorphism

$$(X, A) \to \mathrm{ind}_{c(g)\colon G\to G}(X, A)$$

which takes x to $(1, g^{-1}x)$.

Definition 4.1.6. An *equivariant cohomology theory $H^*_?$ with values in Λ-modules* consists of a G-cohomology theory H^*_G for every group G together with a so-called *induction structure*: For every group homomorphism $\alpha\colon H \to G$ and H-CW-pair (X, A) such that $\ker(\alpha)$ acts freely on X, there exists a natural isomorphism

$$\mathrm{ind}_\alpha\colon H^n_G(\mathrm{ind}_\alpha(X, A)) \xrightarrow{\cong} H^n_H(X, A)$$

for every $n \in \mathbb{Z}$, such that the following conditions are fulfilled:

(i) (Compatibility with the boundary homomorphism) $\delta^n_H \circ \mathrm{ind}_\alpha = \mathrm{ind}_\alpha \circ \delta^n_G$ for all $n \in \mathbb{Z}$.

(ii) (Functoriality) If $\beta\colon G \to K$ is another group homomorphism such that $\ker(\beta \circ \alpha)$ acts freely on X, then for all $n \in \mathbb{Z}$ we have

$$\mathrm{ind}_{\beta\circ\alpha} = \mathrm{ind}_\alpha \circ \mathrm{ind}_\beta \circ H^n_K(f_1)$$

where $f_1\colon \mathrm{ind}_\beta(\mathrm{ind}_\alpha(X, A)) \to \mathrm{ind}_{\beta\circ\alpha}(X, A)$ is the natural K-homeomorphism given by $(k, g, x) \mapsto (k\beta(g), x)$.

(iii) (Compatibility with conjugation) Let $g \in G$ and let $c(g)\colon G \to G$ be the conjugation homomorphism $g' \mapsto gg'g^{-1}$. If (X, A) is a G-CW-pair, then for all $n \in \mathbb{Z}$, the homomorphism

$$\mathrm{ind}_{c(g)\colon G\to G}\colon H^n_G(\mathrm{ind}_{c(g)\colon G\to G}(X, A)) \to H^n_G(X, A)$$

agrees with $H^n_G(f_2)$ where f_2 is the G-homeomorphism

$$(X, A) \to \mathrm{ind}_{c(g)\colon G\to G}(X, A)$$

which takes x to $(1, g^{-1}x)$.

Lemma 4.1.7 ([Lüc05, Lemma 1.1])**.** *Let H, K be finite subgroups of G and $g \in G$ an element with $gHg^{-1} \subset K$ and let $R_{g^{-1}}\colon G/H \to G/K$ be the G-map that takes $g'H$ to $g'g^{-1}K$ and $c(g)\colon H \to K$ the homomorphism that takes h to ghg^{-1}. Let*

$$pr\colon (\mathrm{ind}_{c(g)\colon H\to K}\{*\}) \to \{*\}$$

be the projection to the one-point space. Then the diagram

$$\begin{tikzcd} H^n_G(G/K) \arrow[r, "H^n_G(R_{g^{-1}})"] \arrow[d, "\mathrm{ind}^G_K"', "\cong"] & H^n_G(G/H) \arrow[d, "\mathrm{ind}^G_H"', "\cong"] \\ H^n_K(\{*\}) \arrow[r, "\mathrm{ind}_{c(g)} \circ H^n_K(pr)"] & H^n_H(\{*\}) \end{tikzcd}$$

commutes. □

Remark 4.1.8. Note that for the most relevant examples one always has a homomorphism ind_α, but the condition that $\ker(\alpha)$ acts freely is needed to ensure that ind_α is bijective.

Examples for equivariant homology theories are the Borel construction (see [KL05, Example 20.8]) or equivariant bordism (of smooth manifolds with orientation preserving cocompact proper smooth G-action, see [KL05, Example 20.9]). Example for equivariant cohomology theories are Borel cohomology or equivariant (topological complex) K-theory (see Examples 1.6 and 1.7 of [Lüc05]).

Now one wants to construct equivariant homology and cohomology theories from spectra as for the non-equivariant situation. In Section 20.4 of [KL05] an equivariant homology theory is constructed from a covariant functor from the category of groupoids to the category of spectra. In [Lüc05] an equivariant cohomology theory is constructed from a contravariant functor from the category of groupoids to the category of Ω-spectra. We will introduce these constructions here. First a G-homology theory can be constructed from a covariant $\mathrm{or}(G)$-spectrum:

Lemma 4.1.9 ([KL05, Lemma 20.12])**.** *Let E be a covariant* $\mathrm{or}(G)$*-spectrum. Then a G-homology theory $H^G_*(-;E)$ is given by*

$$H^G_n(X,A;E) = \pi_n(\mathrm{Map}_G(-,(X_+ \cup_{A_+} \mathrm{cone}(A_+))) \wedge_{\mathrm{or}(G)} E).$$

In particular one has $H^G_n(G/H;E) = \pi_n(E(G/H))$. We call this G-homology theory the associated G-homology theory to the covariant $\mathrm{or}(G)$-spectrum E. □

For a contravariant $\mathrm{or}(G)$-Ω-spectrum E we analogously get an associated G-cohomology theory (on pairs of contravariant $\mathrm{or}(G)$-CW-complexes (X,A)) by defining

$$H^n_G(X,A;E) := \pi_{-n}(\mathrm{Map}_{\mathrm{or}(G)}(X^?_+ \cup_{A^?_+} \mathrm{cone}(A^?_+), E)).$$

If one composes the $H^n_G(-;E)$ with the fixed point functor, one gets an associated G-cohomology theory on G-CW-pairs. See [DL98, Section 4] and [Lüc05, Example 1.8] for details.

Let GROUPOIDS be the category of small groupoids. The morphisms of this category are the functors between groupoids. We define $\mathrm{GROUPOIDS}^{\mathrm{inj}}$ to be the subcategory which contains all groupoids but only the functors between groupoids which are faithful, that is they consist of injective maps of the morphism sets.

Next there is a functor from G-sets to $\mathrm{GROUPOIDS}^{\mathrm{inj}}$ which is defined as follows:

Definition 4.1.10. Let S be a G-set. Let $\overline{S}$ be the category whose objects are the elements of the set S and the morphisms between two elements $s_1, s_2 \in S$ are defined by

$$\mathrm{Mor}(s_1, s_2) := \{g \in G \mid gs_1 = s_2\}.$$

Obviously $\overline{S}$ is a groupoid; it is called the *transport groupoid* of S. A G-map between G-sets $S \to T$ defines a functor

$$\begin{array}{rcl} \overline{S} & \to & \overline{T} \\ \text{on objects: } s & \mapsto & f(s) \\ \text{on morphisms: } g & \mapsto & g. \end{array}$$

Therefore this defines a functor from G-sets to GROUPOIDS$^{\mathrm{inj}}$ which is called the *transport groupoid functor.*

Definition 4.1.11. Let G be a group. We define

$$R^G \colon \mathrm{or}(G) \to \mathrm{GROUPOIDS}^{\mathrm{inj}}$$

to be the composition of the transport groupoid functor with the forgetful functor from or(G) to the category of G-sets.

Remark 4.1.12. Let $H \subset G$ be a subgroup. Then there is an equivalence of the categories $\mathcal{H} = \mathrm{or}(H, \{e\})$ and $\overline{G/H}$: The functor

$$\begin{array}{rcl} \mathcal{H} & \to & \overline{G/H} \\ \text{on objects: } * & \mapsto & [e] \\ \text{on morphisms: } h & \mapsto & h \end{array}$$

is essentially surjective, full and faithful. In particular $\overline{G/H}$ is a small category with exactly one isomorphism class of objects.

Lemma 4.1.13 ([KL05, Lemma 20.14])**.** *Let*

$$E \colon \mathrm{GROUPOIDS}^{\mathrm{inj}} \to \mathcal{S}$$

be a covariant GROUPOIDS$^{\mathrm{inj}}$*-spectrum. Suppose that E respects equivalences, that is an equivalence of groupoids (which is an equivalence of the underlying categories) is taken to a weak equivalence of spectra. Then E defines an equivariant homology theory $H_*^?(-;E)$ such that for every group G the G-homology theory is the associated G-homology theory of the covariant* or(G)*-spectrum $E \circ R^G$, that is*

$$H_*^G(X, A; E) = H_*^G(X, A; E \circ R^G).$$

In particular one has

$$H_n^G(G/H; E) \cong H_n^H(\{*\}; E) \cong \pi_n(E(\mathcal{H})).$$

The whole construction is natural in E. ☐

Lemma 4.1.14. *Let*

$$E\colon \text{GROUPOIDS}^{\text{inj}} \to \Omega\text{-}\mathcal{S}$$

be a contravariant $\text{GROUPOIDS}^{\text{inj}}$*-*$\Omega$*-spectrum. Suppose that E respects equivalences, that is an equivalence of groupoids (which is an equivalence of the underlying categories) is taken to a weak equivalence of spectra. Then E defines an equivariant cohomology theory $H^*_?(-;E)$ such that for every group G the G-cohomology theory is the associated G-cohomology theory of the contravariant* $\operatorname{or}(G)$*-spectrum $E \circ R^G$, that is*

$$H^*_G(X, A; E) = H^*_G(X, A; E \circ R^G).$$

In particular one has

$$H^n_G(G/H; E) \cong H^n_H(\{*\}; E) \cong \pi_{-n}(E(\mathcal{H})).$$

The whole construction is natural in E.

Proof. A proof (and so a construction of the induction homomorphism) is given in [Lüc05, Example 1.8]. □

Remark 4.1.15. The proofs of the last two lemmas only need, that E is a functor on the full subcategory of $\text{GROUPOIDS}^{\text{inj}}$ consisting of those groupoids that are transport groupoids of homogeneous spaces of the form G/H. These groupoids have the property that there is exactly one isomorphism class of objects. Therefore one can replace the category $\text{GROUPOIDS}^{\text{inj}}$ by the full subcategory of groupoids with exactly one isomorphism class of objects, which we will denote by $\text{GROUPOIDS}^{\text{inj}}_1$. Thus both lemmas stay true for $\text{GROUPOIDS}^{\text{inj}}_1$-($\Omega$-)spectra that take equivalences to weak equivalences.

4.2 The $\operatorname{or}(G)$-spectra and the equivariant cohomology theory associated to an ad theory with functorial gluing and cylinder constructions

If $\mathcal{A}$ is the target category of an ad theory with functorial gluing and cylinder constructions, then a group homomorphism $G_1 \to G_2$ induces a functor $\mathcal{A}^{\mathcal{G}_2} \to \mathcal{A}^{\mathcal{G}_1}$ which is a 0-morphism and preserves ads. This defines a contravariant functor from the category of groups to the category of ad theories and the composition with the functor from ad theories to Ω-spectra or the functor from ad theories to symmetric spectra defines a contravariant functor from groups to Ω-spectra or to symmetric spectra.

This also holds for the generalization to a larger target category of section 3.5. And it is true for the special case of subcategories of section 3.6 where $\mathcal{A}$ is a topological category and $\mathcal{W}_{\mathcal{H}}$ denotes the subcategory of continuous functors from the topological category $\mathcal{H}$ of a topological group H to $\mathcal{D}_{\mathcal{A}}$ and the group homomorphism is a continuous group homomorphism. Then it defines a functor from the category of topological groups (with continuous group homomorphisms) to Ω-spectra or symmetric spectra respectively.

The aim of this section is to get an equivariant cohomology theory from this data. To get this we want to construct contravariant $\operatorname{or}(G)$-Ω-spectra and use the techniques and results of Section 4.1. In particular we will use Remark 4.1.15 and define a contravariant $\text{GROUPOIDS}^{\text{inj}}_1$-$\Omega$-spectrum which takes equivalences to weak equivalences.

From now on we assume that $\mathcal{A}$ is the target category of an ad theory with functorial gluing and cylinder constructions and all groups that occur will be discrete groups.

The naive approach would be to assign to G/H the Quinn spectrum of the ad theory with target category $\mathcal{A}^{\mathcal{H}}$. Unfortunately this definition fails for morphisms: Recall from Proposition 6.3.4 that all the morphisms $G/H \to G/K$ in the orbit category are of the form R_a for an $a \in G$ with $a^{-1}Ha \subset K$. Furthermore $R_a = R_b$ if and only if $ab^{-1} \in K$. Therefore assigning the group homomorphism $H \to K, h \mapsto a^{-1}ha$ to R_a is not well-defined.

We will use the solution of this problem that is given in [DL98]. Here it is important, that the index category $\mathcal{C}$ does not only have to be the category of a group, it also can be any small category with exactly one isomorphism class of objects. In particular it can be the transport groupoid of a homogeneous space of the form G/H. And we have seen that a functor (resp. equivalence) between small categories with exactly one isomorphism class induces a morphism (resp. equivalence) of the associated ad theories. Such an equivalence of ad theories induces isomorphisms between the bordism groups of the ad theories by Remark 2.2.7. The bordism groups are the homotopy groups of the spectrum associated to the ad theory. This proves the following theorem:

Theorem 4.2.1. *Let $\mathcal{A}$ be the target category of an ad theory $\mathrm{ad}_{\mathcal{A}}$ with functorial gluing and cylinder constructions. Let $R_{\mathcal{A}}$ be the contravariant functor from $\mathrm{GROUPOIDS}_1^{\mathrm{inj}}$ to the category of ad theories which assigns to such a groupoid $\mathcal{C}$ the associated ad theory $\mathrm{ad}_{\mathcal{A}^{\mathcal{C}}}$ and let Q denote the functor from ad theories to Ω-spectra. Then the composition*

$$Q \circ R_{\mathcal{A}} \colon \mathrm{GROUPOIDS}_1^{\mathrm{inj}} \to \Omega\text{-}\mathcal{S}$$

defines a contravariant $\mathrm{GROUPOIDS}_1^{\mathrm{inj}}$-$\Omega$-spectrum which takes equivalences to weak equivalences of spectra. □

For each group G we denote by R^G the functor from or(G) to $\mathrm{GROUPOIDS}_1^{\mathrm{inj}}$ which assigns to G/H its transport groupoid. Then the composition $Q \circ R_{\mathcal{A}} \circ R^G$ defines a contravariant or(G)-Ω-spectrum.

By Remark 4.1.15 and Lemma 4.1.14 we get the following corollary:

Corollary 4.2.2. *Let $\mathcal{A}$ be the target category of an ad theory with functorial gluing and cylinder constructions. Then the contravariant $\mathrm{GROUPOIDS}_1^{\mathrm{inj}}$-$\Omega$-spectrum $Q \circ R_{\mathcal{A}}$ of Theorem 4.2.1 defines an equivariant cohomology theory $H_?^*(-; Q \circ R_{\mathcal{A}})$ such that for every group G the G-cohomology theory is the G-cohomology theory associated to the contravariant or(G)-spectrum $Q \circ R_{\mathcal{A}} \circ R^G$, that is*

$$H_G^*(X, A; Q \circ R_{\mathcal{A}}) = H_G^*(X, A; Q \circ R_{\mathcal{A}} \circ R^G).$$

In particular one has

$$H_G^n(G/H; Q \circ R_{\mathcal{A}}) \cong H_H^n(\{*\}; Q \circ R_{\mathcal{A}}) \cong \pi_{-n}(Q \circ R_{\mathcal{A}}(\mathcal{H})) \cong \Omega_{-n}(\mathrm{ad}_{\mathcal{A}^{\mathcal{H}}}).$$

The whole construction is natural in the ad theory with functorial gluing and cylinder constructions.

Proof. It remains to show that the construction is natural in ad theories with functorial gluing and cylinder constructions, but this is a direct consequence of the fact that a morphism of such ad theories $\mathcal{A} \to \mathcal{A}'$ induces a morphism of the associated ad theories $\mathcal{A}^{\mathcal{C}} \to \mathcal{A}'^{\mathcal{C}}$ for a small category $\mathcal{C}$ with exactly one isomorphism class of objects and that this defines a functor (see Theorem 3.3.1). $\square$

Remark 4.2.3. What we have done in this section generalizes to the situation of Section 3.5: For a fixed ad theory with target category $\mathcal{A}$ with functorial gluing and cylinder constructions with respect to $\mathcal{D}$, we replace the functor $R_{\mathcal{A}}$ by the functor $R_{\mathcal{D}_{\mathcal{A}}}$ which assigns to a groupoid $\mathcal{C}$ (with one isomorphism class of objects) the associated ad theory with target category $\mathcal{D}_{\mathcal{A}}^{\mathcal{C}}$. Then the theorem and the corollary hold as well (if we simply ignore the naturality stated in the last sentence of the corollary).

5 Examples

In this part we give short descriptions of each of the standard examples for ad theories, which are the ad theory of a chain complex, the ad theory of oriented topological bordism, geometric Poincaré complexes, symmetric Poincaré ad theories and quadratic Poincaré ad theories.

We then show, that all these ad theories are ad theories with functorial gluing and cylinder constructions. We give examples for the new ad theories and equivariant (co)homology theories we get.

5.1 The ad theory of a chain complex

Let C be a chain complex. The $\mathbb{Z}$-graded category $\mathcal{A}_C$ associated to a chain complex was introduced in Section 2.1.

Example 5.1.1. If K is a ball complex then let $\mathrm{cl}(K)$ denote its cellular chain complex. The abelian group $\mathrm{cl}_n(K)$ is generated by the symbols $\langle\sigma, o\rangle$ with σ an n-dimensional cell with orientation o, subject to the relation given by $\langle\sigma, -o\rangle = -\langle\sigma, o\rangle$. Then a pre K-ad F lifts to a map of graded abelian groups from $\mathrm{cl}(K) \to C$. One defines F to be a K-ad if and only if this lift is a chain map. By part a) of the definition of ad theories, a (K, L)-ad has to be a K-ad which is zero on $\mathcal{C}\mathrm{ell}(L)$. Gluing is given by addition and $J(F)$ is defined to be 0 on all the objects of $K \times I$ that are not contained in $K \times 0$ or $K \times 1$. This defines an ad theory ad_C associated to a chain complex.

Note that a $*$-ad is a cycle of C. Then by definition we have a bijection between $H_k C$ and the bordism group Ω_k. From the construction of the addition in the bordism groups and the fact that gluing is addition, one sees that this bijection is an isomorphism of abelian groups.

If C is a DGA (see [Wei94, p. 112] for a definition), then ad_C is a multiplicative ad theory.

We show that this ad theory fulfills the requirements of theorem 3.3.1, namely it has functorial gluing and cylinder constructions:

Proposition 5.1.2. *For a chain complex C the ad theory ad_C is an ad theory with functorial gluing and it has functorial cylinders.*

Proof. Recall that the gluing construction was given by addition. Thus we define the functor G to be addition on objects. Regarding the definition of $\mathcal{A}_C$ we see that there are no morphisms between objects of the same dimension other than the identity. Therefore natural transformations between k-morphisms can only be the identity. Hence G already defines a functor which is the identity functor on residual subcomplexes.

For the functorial cylinders recall that the natural transformation J was defined to be 0 on objects that are not contained in $K \times 1$ or $K \times 0$. Again there are no natural transformations other than the identity between K-ads of degree k, so $J(K)$ already defines a functor and $J(K)$ is the identity functor on $K \times 0$ and $K \times 1$ because it is the identity on objects. $\square$

The proof shows that this example is rather trivial. There is no interesting structure which can be described by diagrams in $\mathcal{A}_C$. One should regard the proof above as a check for consistency.

5.2 Balanced categories and functors

For the other examples we need the notion of a balanced category which is introduced in Section 5 of [LM09]. Let $\mathcal{A}$ be a $\mathbb{Z}$-graded category and let $\mathcal{A}(A, B)$ denote the set of morphisms in $\mathcal{A}$ from A to B.

Definition 5.2.1. $\mathcal{A}$ is called a *balanced category*, if there is a natural bijection

$$\eta\colon \mathcal{A}(A, B) \to \mathcal{A}(A, i(B))$$

for objects A, B with $\dim A < \dim B$, such that

(a) $\eta \circ i = i \circ \eta$

(b) $\eta \circ \eta$ is the identity.

A functor F between balanced categories is called a *balanced functor* if it commutes strictly with η: $F \circ \eta = \eta \circ F$.

Example 5.2.2. All the $\mathbb{Z}$-graded categories mentioned so far are balanced. In particular the categories $\mathcal{C}\mathrm{ell}(K, L)$ are balanced.

Definition 5.2.3. If $\mathcal{A}$ is a balanced category, then a *balanced pre* (K, L)*-ad* with values in $\mathcal{A}$ is a pre (K, L)-ad which is a balanced functor.

5.3 Oriented topological bordism

Now we can present the ad theory of oriented topological bordism, introduced in Section 6 of [LM09]. Its target category will be $\mathcal{A}_{STop}$. Let $\mathcal{B}$ be the category of compact orientable topological manifolds, whose dimension-preserving morphisms are the orientable inclusions (that is there exists some choice of orientations which is preserved by them) and the morphisms which increase dimension are the inclusions with image in the boundary.

For a ball complex K let $\mathcal{C}\mathrm{ell}^\flat(K)$ be the category whose objects are the cells of K together with an empty cell in every dimension and whose morphisms are the inclusions of cells. Then a balanced pre K-ad F with values in $\mathcal{A}_{STop}$ induces a functor

$$F^\flat\colon \mathcal{C}\mathrm{ell}^\flat(K) \to \mathcal{B}.$$

Let σ, σ' be cells of K with $\sigma' \subsetneq \sigma$. Write $i_{(\sigma',o'),(\sigma,o)}$ for the morphism $(\sigma', o') \to (\sigma, o)$ in $\mathcal{C}\mathrm{ell}(K)$ and $j_{\sigma',\sigma}$ for the morphism in $\mathcal{C}\mathrm{ell}^\flat(K)$ from σ' to σ.

Definition 5.3.1. A K-ad with values in $\mathcal{A}_{STop}$ is a balanced pre K-ad F of degree k such that

(a) If (σ', o') and (σ, o) are oriented cells with $\dim\sigma' = \dim\sigma - 1$ and if $[o, o']$ is equal to $(-1)^k$ then the map

$$F(i_{(\sigma',o'),(\sigma,o)})\colon F(\sigma', o') \to \partial F(\sigma, o)$$

is orientation-preserving.

(b) For each σ, $\partial F^\flat(\sigma)$ is the colimit in Top of $F^\flat|_{\mathcal{C}\mathrm{ell}^\flat(\partial\sigma)}$.

The set of K-ads of degree k is denoted by $\mathrm{ad}^k_{STop}(K)$.

Example 5.3.2. The functor $\mathcal{C}\mathrm{ell}(\Delta^n) \to \mathcal{A}_{STop}$ which takes each oriented cell of Δ^n to itself as an oriented topological manifold is an ad.

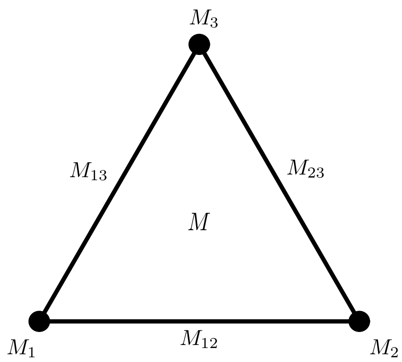


Figure 3: A Δ^2-ad.

Theorem 5.3.3 ([LM09, Theorem (16.5)]). ad_{STop} *is an ad theory.* □

The sign in part (a) of the definition is needed to prove the reindexing axiom. We give a detailed description of gluing and the construction of cylinder ads in the proof of Proposition 5.3.4 later. For the moment one only has to know, that gluing is done by taking the colimit of the underlying manifolds. This, together with the description of the addition of the bordism groups shows, that the addition is given by taking the disjoint union. Thus the bordism groups of ad_{STop} are the usual oriented topological bordism groups.

If one redefines the term topological manifold to mean a topological manifold which is a subspace of some $\mathbb{R}^n$, then $\mathcal{A}_{STop}$ has a strict monoidal structure given by the Cartesian product. This makes ad_{STop} a multiplicative ad theory.

We proceed to examine the functoriality of the gluing and cylinder constructions of these ad theory. In addition to $\mathcal{A}_{STop}$ we will need the $\mathbb{Z}$-graded category $\mathcal{D}_{STop}$ (see

Definition 2.1.6). We will denote these categories by $\mathcal{A}$ and $\mathcal{D}$ respectively here and take into account that we are in the situation of Section 3.5: The category $\mathcal{A}_{STop}$ is a $\mathbb{Z}$-graded subcategory of the $\mathbb{Z}$-graded category $\mathcal{D}_{STop}$ and it contains all objects of $\mathcal{D}$.

Furthermore if we redefine the term *topological manifold* to mean a topological manifold which is a subset of an Euclidean space, then $\mathcal{D}_{STop}$ has a strict monoidal structure as a $\mathbb{Z}$-graded category, and this strict monoidal structure restricts to the strict monoidal structure on the $\mathbb{Z}$-graded subcategory $\mathcal{A}_{STop}$. It is given by the Cartesian product of oriented topological manifolds.

Proposition 5.3.4. *The ad theory* ad_{STop} *of oriented topological manifolds is an ad theory with functorial gluing and cylinder constructions with respect to* $\mathcal{D} = \mathcal{D}_{STop}$.

Proof. First we show that gluing is functorial. We will give a detailed description of the gluing construction for oriented topological manifolds and then show that it is functorial. Let K' be a subdivision of K. For a K'-ad F of degree k we denote the gluing construction of the proof of Theorem (6.5) of [LM09] by $G(F)$. Explicitly the induced functor $G(F)^\flat$ is given by Proposition (6.6) of [LM09], that is for a cell σ it is the orientable topological manifold

$$G(F)^\flat(\sigma) := \operatorname*{colim}_{\tau \in K',\, \tau \subset \sigma} F^\flat(\tau).$$

For an orientation o of σ there is exactly one orientation $\tilde{o}$ of $G(F)^\flat(\sigma)$ such that for every oriented cell (τ, o') with $\tau \subset \sigma$, $\tau \in K'$, $\dim \tau = \dim \sigma$, and $i_{(\tau,o'),(\sigma,o)}$ an orientation preserving inclusion of cells $(\tau, o') \to (\sigma, o)$, the inclusion of manifolds $F(\tau, o') \subset G(F)^\flat(\sigma)$ is orientation-preserving. Then $G(F)$ is defined on (σ, o) to be $G(F)^\flat(\sigma)$ equipped with this orientation $\tilde{o}$. In particular this orientation ensures that $G(F)$ fulfills part a) of Definition 5.3.1 (part b) is fulfilled by Proposition (6.6) of [LM09]).

For a morphism $j\colon (\sigma', o') \to (\sigma, o)$ in $\mathcal{C}\mathrm{ell}(K)$ (which is an inclusion of cells) $G(F)(j)$ is defined to be the induced map between the colimits. The colimit of inclusions is an inclusion, so $G(j)$ is a morphism in $\mathcal{A}_{STop}$. Furthermore $G(F)\colon \mathcal{C}\mathrm{ell}(K) \to \mathcal{A}_{STop}$ is a functor, because colimit is a functor. It is a K-ad of degree k.

Now let $g\colon F_1 \to F_2$ be a morphism in the category $\mathrm{ad}^{k,\mathcal{D}}(K')$. Thus it is a natural transformation of the K'-ads composed with the inclusion of $\mathcal{A}_{STop}$ into $\mathcal{D}_{STop}$, that is it consists of morphisms $g_{(\sigma,o)}\colon F_1(\sigma, o) \to F_2(\sigma, o)$ in $\mathcal{D}_{STop}$ such that for morphisms $(\sigma', o') \to (\sigma, o)$ in $\mathcal{C}\mathrm{ell}(K')$ the diagrams

$$\begin{tikzcd}
F_1(\sigma, o) \arrow[r] & F_2(\sigma, o) \\
F_1(\sigma', o') \arrow[r] \arrow[u] & F_2(\sigma', o') \arrow[u]
\end{tikzcd}$$

commute. We define $G(g)_{(\sigma,o)}$ to be the colimit of the $g_{(\tau,o')}$ of all cells $\tau \in K'$ with $\tau \in \sigma$. This defines a continuous map between oriented topological manifolds which is orientation-preserving if the dimensions of source and target agree, because then the maps $g_{(\tau,o')}$ were orientation-preserving. If the dimensions do not agree it is map with image in the boundary because Proposition (6.6) preserves that. Taking colimit is a functor, so $G(g)$ defines a morphism in $\mathrm{ad}^{k,\mathcal{D}}(K)$. The same argument shows that with these definitions G is a functor $\mathrm{ad}^{k,\mathcal{D}}(K') \to \mathrm{ad}^{k,\mathcal{D}}(K)$. On residual subcomplexes this functor restricts to the identity functor.

We proceed to show that the cylinder constructions of ad_{STop} are functorial with respect to $\mathcal{D}_{STop}$. Let K be a ball complex. The cylinder construction was done as follows: If (σ, o) is an oriented cell, then $J(F)(\sigma \times I, o \times o') = F(\sigma, o) \times (I, o')$.

Again a morphism $g\colon F_1 \to F_2$ in $\mathrm{ad}^{k,\mathcal{D}}(K)$ is given by maps $g_{(\sigma,o)}\colon F_1(\sigma, o) \to F_2(\sigma, o)$ in $\mathcal{D}_{STop}$ such that for morphisms $(\sigma', o') \to (\sigma, o)$ the diagrams above commute. Thus we simply define $J(K)(g)(\sigma \times I, o \times o') := g_{(\sigma,o)} \times \mathrm{id}_{(I,o')}$. This map is orientation-preserving if $g_{(\sigma,o)}$ was orientation-preserving so it is a morphism in $\mathcal{D}_{STop}$. It is clear that this defines a morphism in $\mathrm{ad}^{k,\mathcal{D}}(K \times I)$ and that with this definition $J(K)$ is a functor $\mathrm{ad}^{k,\mathcal{D}}(K) \to \mathrm{ad}^{k,\mathcal{D}}(K \times I)$, that it restricts to the identity functor on $K \times 0$ and $K \times 1$ and that it takes trivial ads to trivial ads. □

This proposition allows us to apply Theorem 3.5.11:

Corollary 5.3.5. *For every small category $\mathcal{C}$ with exactly one isomorphism class of objects we get an associated ad theory with target category $\mathcal{D}^{\mathcal{C}}_{\mathcal{A}}$ where $\mathcal{D} = \mathcal{D}_{STop}$ and $\mathcal{A} = \mathcal{A}_{STop}$.* □

Because of the strict monoidal structures we can also apply theorem 3.5.13 and we get

Corollary 5.3.6. *This associated ad theory is a multiplicative ad theory.* □

Example 5.3.7 (Topological oriented G-bordism). Let $\mathcal{G}$ be the category of a discrete group G and let $*$ be the only object of this category. Thus it is a small category with only one object. Then $\mathcal{D}^{\mathcal{G}}_{\mathcal{A}}$ is the category of oriented topological G-manifolds with oriented equivariant inclusions for manifolds of the same dimension and with equivariant inclusions with image in the boundary for manifolds with different dimensions.

From the definition of bordism groups and the construction of the addition in the bordism groups of the ad theory together with the gluing construction, we can see that the bordism groups of this ad theory are the bordism groups of oriented topological G-bordism. This is completely analogous to Remark (6.7) in [LM09].

The strict monoidal structure on the category $\mathcal{D}^{\mathcal{G}}_{\mathcal{A}}$ is given by the product of G-manifolds with the diagonal action. The unit object is the point together with the trivial action.

Remark 5.3.8. It is remarkable, that this ad theory, and therefore the associated spectra really represent G-bordism, since for the equivariant Thom spectrum this is generally not true, due to a failure of transversality in the equivariant situation. An example for this failure is given by the manifold $\mathbb{R}$ as a $G := \mathbb{Z}/2\mathbb{Z}$-manifold with the only non-trivial linear action. Then the inclusion of the (trivial) G-manifold given by a point $*$ to 0 is the only G-map from $*$ to $\mathbb{R}$. In particular it can not be homotopic to a G-map that would take $*$ to a point different from 0, which would be the only possibility for the inclusion of a point to be transverse to the submanifold $\{0\}$. This failure of transversality is the problem that prevents the definition of an inverse of the Pontryagin-Thom construction.

After all this we can apply the results of Section 4.2. Recall that Q denotes the functor from ad theories to Ω-spectra and that $R_{\mathcal{D}_\mathcal{A}}$ is the functor which assigns to a groupoid $\mathcal{C}$ with exactly one isomorphism class of objects the associated ad theory with target category $\mathcal{D}_\mathcal{A}^\mathcal{C}$.

Corollary 5.3.9. *We get an equivariant cohomology theory $H^*_?(-; Q \circ R_{\mathcal{D}_\mathcal{A}})$ such that for every group G the G-cohomology theory is the G-cohomology theory associated to the contravariant* $\mathrm{or}(G)$*-spectrum $Q \circ R_{\mathcal{D}_\mathcal{A}} \circ R^G$. In particular we have*

$$H^n_G(G/H; Q \circ R_{\mathcal{D}_\mathcal{A}}) \cong \Omega_{-n}(\mathrm{ad}_{\mathcal{D}_\mathcal{A}^\mathcal{H}}),$$

which are the bordism groups of H-bordism of oriented topological H-manifolds. □

Example 5.3.10 (Bordism of homeomorphisms)*.* Note that the discrete group G does not have to be finite. For $G = \mathbb{Z}$ one gets an ad theory with values in $\mathbb{Z}$-manifolds. A topological $\mathbb{Z}$-manifold is the same as a topological manifold with homeomorphism, which can be defined analogously to manifolds with diffeomorphism (see [Kre84]). Then the bordism groups of this ad theory are the bordism groups of manifolds with homeomorphisms.

We proceed by giving some applications of the generalization to subcategories of Section 3.6.

Example 5.3.11 (Restricted isotropy)*.* Let G be a discrete groups and $\mathcal{F}$ a family of subgroups closed under taking subgroups and conjugation. Let $\mathcal{W}_\mathcal{F}$ be the full subcategory of $\mathcal{D}_\mathcal{A}^\mathcal{G}$ consisting of the G-manifolds M with isotropy restricted to $\mathcal{F}$, that is for each $x \in M$ the isotropy group of x is in $\mathcal{F}$. It is clear, that the involution restricts to $\mathcal{W}_\mathcal{F}$ and that the image of $\emptyset$ is in $\mathcal{W}_\mathcal{F}$.

Gluing together G-manifolds along inclusions of G-manifolds and taking the product of a G-manifold with I (equipped with the trivial action) and giving the result the diagonal action both preserves the isotropy groups. Therefore we can apply Theorem 3.6.2 and we get an ad theory with values in $\mathcal{W}_\mathcal{F}$.

The bordism groups of this ad theory are the G-bordism groups of oriented topological G-manifolds with isotropy in $\mathcal{F}$.

For example if $\mathcal{F}$ is the family consisting only of the trivial subgroup given by the neutral element, we obtain an ad theory with values in free G-manifolds whose bordism groups are the bordism groups of G-bordism of free G-manifolds.

Example 5.3.12 (Continuous actions)*.* Let G be a topological group. In particular the category $\mathcal{G}$ of the group is a topological category. Note that $\mathcal{D} = \mathcal{D}_{STop}$ is a topological category as well and that continuous G-actions on oriented topological manifolds are continuous functors to $\mathcal{D}_{STop}$. More generally let $\mathcal{C}$ be a small topological category with discrete set of objects and exactly one isomorphism class of objects. Define $\mathcal{W}$ to be the the full subcategory of $\mathcal{D}_\mathcal{A}^\mathcal{C}$ of the continuous functors.

We know, that the gluing construction is given by taking the colimit of the $\mathcal{C}$-manifolds over a subdivided cell to get a new $\mathcal{C}$-manifold. Manifolds are locally compact and Hausdorff, hence Proposition 6.1.2 ensures that the gluing construction of section 3.5

restricts to $\mathcal{W}$: If F is a K'-ad for a subdivision K' of a ball complex K with values in $\mathcal{W}$, then for pairs (C_1, C_2) of objects of $\mathcal{C}$ and for a cell σ of K the continuity of the maps

$$\mathrm{Map}(C_1, C_2) \to \mathrm{Map}(F(\tau, o')(C_1), F(\tau, o')(C_2)), \quad g \mapsto F(\tau, o')(g)$$

for every cell $\tau \in K', \tau \subset \sigma$ implies that the colimit map

$$\mathrm{Map}(C_1, C_2) \to \mathrm{Map}(\operatorname*{colim}_{\tau \in K', \tau \subset \sigma} F^\flat(\tau)(C_1), \operatorname*{colim}_{\tau \in K', \tau \subset \sigma} F^\flat(\tau)(C_2))$$

is continuous.

The cylinder construction is given by taking the (value-wise) product of a $\mathcal{C}$-manifold with the constant $\mathcal{C}$-space I. Thus Proposition 6.1.4 ensures that the cylinder construction of section 3.5 restricts to $\mathcal{W}$: For a K-ad F with values in $\mathcal{W}$ and pairs (C_1, C_2) of objects of $\mathcal{C}$ the continuity of the maps

$$\mathrm{Map}(C_1, C_2) \to \mathrm{Map}(F(\sigma, o)(C_1), F(\sigma, o)(C_2)), \quad g \mapsto F(\sigma, o)(g)$$

implies that the map

$$\mathrm{Map}(C_1, C_2) \to \mathrm{Map}(F^\flat(\sigma)(C_1) \times I, F^\flat(\sigma)(C_2) \times I)$$

is continuous.

Moreover the strict monoidal structure of $\mathcal{D}^{\mathcal{C}}_{\mathcal{A}}$ restricts to continuous functors and therefore we can apply both theorems of section 3.6 and get an ad theory with values in $\mathcal{W}$. If $\mathcal{C} = \mathcal{G}$ is the topological category of a topological group G, then the bordism groups are the oriented topological G-bordism groups.

Example 5.3.13. The last two examples can be combined to get an ad theory with values in oriented topological G-manifolds with restricted isotropy for a topological group G.

Now we return to the situation where G is a discrete group. Let X be a G-space, that is it is a functor $X\colon \mathcal{G} \to \mathrm{Top}$. Then we can define an ad theory of singular G-manifolds in X as follows: Let $\mathcal{D}^{\mathcal{G}}_{\mathcal{A}}(X)$ denote the category whose objects are the natural transformations from objects of $\mathcal{D}^{\mathcal{G}}_{\mathcal{A}}$ to X (so they are G-maps from G-manifolds to X). The morphisms are morphisms of $\mathcal{D}^{\mathcal{G}}_{\mathcal{A}}$ (which are natural transformations representing equivariant inclusions between G-manifolds) which commute with these natural transformations to X. So the morphisms are equivariant inclusions of G-manifolds, which commute with the G-maps to X (and have to be orientation-preserving if the dimensions of the G-manifolds agree). The dimension of an object of $\mathcal{D}^{\mathcal{G}}_{\mathcal{A}}$ is the dimension of the underlying G-manifold. An involution is given by taking a singular G-manifold to the same G-manifold with reversed orientation and the same G-map to X. In every dimension there is an object $\emptyset_n$ consisting of the empty G-manifold of dimension n together with the empty map to X.

Then $\mathcal{D}^{\mathcal{G}}_{\mathcal{A}}(X)$ is a $\mathbb{Z}$-graded category. Note that $\mathcal{D}^{\mathcal{G}}_{\mathcal{A}}(*)$ is equal to $\mathcal{D}^{\mathcal{G}}_{\mathcal{A}}$. A G-map $X_1 \to X_2$ induces a 0-morphism of $\mathbb{Z}$-graded categories

$$\mathcal{D}^{\mathcal{G}}_{\mathcal{A}}(X_1) \to \mathcal{D}^{\mathcal{G}}_{\mathcal{A}}(X_2)$$

and this defines a functor from the category of G-spaces to the category of $\mathbb{Z}$-graded categories.

Definition 5.3.14. A pre K-ad F with values in $\mathcal{D}^{\mathcal{G}}_{\mathcal{A}}(X)$ is an ad if the composition

$$\mathcal{C}\mathrm{ell}(K) \xrightarrow{F} \mathcal{D}^{\mathcal{G}}_{\mathcal{A}}(X) \longrightarrow \mathcal{D}^{\mathcal{G}}_{\mathcal{A}}(*) = \mathcal{D}^{\mathcal{G}}_{\mathcal{A}}$$

is an ad. The set of K-ads of degree k will be denoted by $\mathrm{ad}^{G,k}_{STop,X}$.

Theorem 5.3.15. $\mathrm{ad}^{G}_{STop,X}$ *is an ad theory and this defines a covariant functor from G-spaces to ad theories.*

Proof. For the gluing axiom one applies the gluing construction of the ad theory of G-manifolds and takes the colimit of the maps to X as the new map to X. For the cylinder axiom we take the cylinder construction for G-manifolds and to the product $M \times I$ of a G-manifold M with I we assign the G-map $M \times I \to X$ which one gets by composing the G-map from M to X with the projection to M. All other axioms and the fact that this is a functor are a direct consequence from the definitions. □

Remark 5.3.16. The bordism groups of this ad theory are the G-bordism groups of singular oriented topological G-manifolds in X; they will be denoted by $\Omega^G_k(X)$. This is a direct consequence of the gluing construction.

If we compose this functor with the functor Q from ad theories to Ω-spectra we get a covariant functor from G-spaces to Ω-spectra. The restriction of this functor to the subcategory given by the orbit category $\mathrm{or}(G)$ defines a covariant $\mathrm{or}(G)$-Ω-spectrum. Thus we can apply Lemma 4.1.9 and get a G-homology theory $H^G_*(-; Q \circ \mathrm{ad}^G_{STop,?})$ such that $H^G_n(G/H; Q \circ \mathrm{ad}^G_{STop,?}) = \pi_n(Q \circ \mathrm{ad}^G_{STop,G/H}) \cong \Omega^G_n(G/H)$.

5.4 Geometric Poincaré ad theories

The next example is given by geometric Poincaré ad theories ([LM09, Section 7]). Let π be a group with a fixed properly discontinuous left action of π on a simply connected space Z. Then the projection $Z \to Z/\pi$ is a universal cover. Let $\omega\colon \pi \to \{\pm 1\}$ be a fixed group homomorphism. The right action of π on $\mathbb{Z}$ detemined by ω will be denoted by $\mathbb{Z}^\omega$.

Definition 5.4.1. Let $f\colon X \to Z/\pi$ be a map and let $\widetilde{X}$ be the pullback of the universal cover Z to X. Then define $S_*(X; \mathbb{Z}^f)$ to be the module $\mathbb{Z}^\omega \otimes_{\mathbb{Z}[\pi]} S_*(\widetilde{X})$. Here $S_*(\widetilde{X})$ denotes the singular chain complex of $\widetilde{X}$.

Definition 5.4.2. Let $\mathcal{A}_{\pi,Z,\omega}$ be the following $\mathbb{Z}$-graded category: The objects are triples

$$(X, f\colon X \to Z/\pi, \xi \in S_*(X, \mathbb{Z}^f)),$$

where X is a space that is homotopy equivalent to a finite CW complex. If $\dim \xi < \dim \xi'$ then a morphism $(X, f, \xi) \to (X', f', \xi')$ is a map $g\colon X \to X'$ such that $f' \circ g = f$. If $\dim \xi = \dim \xi'$ one requires additionally that $g_*(\xi) = \xi'$ and there are no morphisms if $\dim \xi > \dim \xi'$. Now the dimension of (X, f, ξ) is $\dim \xi$, the involution i takes (X, f, ξ) to $(X, f, -\xi)$ and $\emptyset_n$ is the n-dimensional object with $X = \emptyset$.

$\mathcal{A}_{\pi,Z,\omega}$ is balanced. A balanced pre K-ad F will be denoted by

$$F(\sigma,o) = (X_\sigma, f_\sigma, \xi_{\sigma,o}),$$

reflecting the fact, that X_σ and f_σ do not depend on o.

Now there are two conditions one wants F to fulfill: X should be well-behaved as a functor from $\mathcal{C}\mathrm{ell}^\flat(K)$ to topological spaces (see [LM09, Definition 7.3]) and F should be closed (see [LM09, Definition 7.6]). For a cell σ of a ball complex K one defines $X_{\partial\sigma}$ to be the space $\mathrm{colim}_{\tau\subsetneq\sigma} X_\tau$. Then the two conditions guarantee, that one can define a cap product using the Alexander-Whitney map (see Definition 7.5):

$$\mathrm{Hom}_{\mathbb{Z}[\pi]}(S_*(\widetilde{X}_\sigma), \mathbb{Z}[\pi]) \to S_*(\widetilde{X}_\sigma)/S_*(\widetilde{X}_{\partial\sigma});$$

and this is a chain map for each σ (see Lemma 7.8). Therefore we can define:

Definition 5.4.3 ([LM09, Definition (7.9)])**.** A pre K-ad F is a K-ad if it is balanced and closed and X is well-behaved and for each (σ,o) the cap product induces an isomorphism

$$H^*(\mathrm{Hom}_{\mathbb{Z}[\pi]}(S_*(\widetilde{X}_\sigma), \mathbb{Z}[\pi])) \to H_{\dim\sigma - \deg F - *}(\widetilde{X}_\sigma, \widetilde{X}_{\partial\sigma}).$$

One writes $\mathrm{ad}_{\pi,Z,\omega}$ for the set of K-ads with values in $\mathcal{A}_{\pi,Z,\omega}$.

Theorem 5.4.4 ([LM09, Theorem (7.10)])**.** $\mathrm{ad}_{\pi,Z,\omega}$ *is an ad theory.* □

Further descriptions of the gluing and cylinder constructions can be found in the proof of this theorem in [LM09] or later in the proof of Proposition 5.4.5 here. Basically, for gluing one takes the colimit of the spaces. Therefore one can see, that addition in the bordism groups of $\mathrm{ad}_{\pi,Z,\omega}$ is given by the disjoint union.

If one replaces the monoidal categories Set and Ab by equivalent strict monoidal categories (see [Kas95, Section XI.5]), the category $\mathcal{A}_{e,*,1}$ has a strict monoidal structure and the ad theory $\mathrm{ad}_{e,*,1}$ is a multiplicative ad theory.

Proposition 5.4.5. *The ad theory of geometric Poincaré bordism ([LM09, section 7]) is an ad theory with functorial gluing and cylinder constructions.*

Proof. Let K' be a subdivision of K and F a K'-ad with $F(\tau,o) = (X_\tau, f_\tau, \xi_{\tau,o})$. We denote the gluing construction given in [LM09] by $G(F)$. We write $V_\sigma := \mathrm{colim}_{\tau\in K'} X_\tau$ and $e_\sigma\colon V_\sigma \to Z/\pi$ for the map induced by the colimit, and $\theta_{\sigma,o}$ for the sum $\sum_{(\tau,o')} \xi_{\tau,o'}$ where (τ,o') runs through the cells of K' with the same dimension as σ and orientation o' induced by o. Then

$$G(F)(\sigma,o) = (V_\sigma, e_\sigma, \theta_{\sigma,o}).$$

Now let $g\colon F_1 \to F_2$ be a natural transformation of K'-ads of degree k, that is it is given by morphisms $g_{(\tau,o)}\colon F_1(\tau,o) \to F_2(\tau,o)$. As above we denote the values $G(F_1)(\sigma,o)$ of $G(F_1)$ and $G(F_2)(\sigma,o)$ of $G(F_2)$ by $(V^1_\sigma, e^1_\sigma, \theta^1_{\sigma,o})$ and $(V^2_\sigma, e^2_\sigma, \theta^2_{\sigma,o})$ respectively. Let $\widetilde{g}_\sigma$ denote the map induced between the colimits by g. Because taking colimit is a functor it is clear that $e^2_\sigma \circ \widetilde{g}_\sigma = e^1_\sigma$. Because of ${g_{(\tau,o')}}_*\xi^1_\tau = \xi^2_\tau$ we get also that $\widetilde{g}_{\sigma*}(\theta^1_{\sigma,o}) = \theta^2_{\sigma,o}$, so $\widetilde{g}_\sigma$ is a morphism in $\mathcal{A}_{\pi,Z,\omega}$. Again because taking colimit is a functor, these maps $\widetilde{g}_{(\sigma,o)} := \widetilde{g}_\sigma$ define a natural transformation $G(g)\colon G(F_1) \to G(F_2)$. The same argument together with the fact that the induced map of the colimit in homology is compatible with the sums of the homology classes of the gluing construction implies that with this definition $G\colon \mathrm{ad}^k_{\pi,Z,\omega}(K') \to \mathrm{ad}^k_{\pi,Z,\omega}(K)$ is a functor. By definition G restricts to the identity functor on residual subcomplexes.

The proof of the functoriality of the cylinder constructions is similar to the proof for ad_{STop}: Just recall from [LM09] that $J(F)$ was defined to be the product ad $F \times G$ of Lemma (7.11) where G is the trivial I-ad with values in $\mathcal{A}_{e,*,1}$, with e the trivial group, and 1 the homomorphism $e \to \{\pm 1\}$. Explicitly $G(\sigma, o) = (\sigma, *, \pm\,\mathrm{id})$ where $\pm$ is $+$ if and only if o is the standard orientation of σ and id denotes the class of the singular chain complex induced by the identity of σ. Therefore if $g\colon F_1 \to F_2$ is a natural transformation of K-ads, the product induces a natural transformation $J(g)\colon J(F_1) \to J(F_2)$ of $(K \times I)$-ads. Of course this defines a functor $J\colon \mathrm{ad}^k_{\pi,Z,\omega}(K) \to \mathrm{ad}^k_{\pi,Z,\omega}(K \times I)$. On $K \times 0$ and $K \times 1$ the functor J restricts to the identity functor and it takes trivial ads to trivial ads by definition. $\square$

As a consequence we get associated ad theories of G-objects in $\mathcal{A}_{\pi,Z,\omega}$ for a group G which we will denote by $\mathrm{ad}^G_{\pi,Z,\omega}$. The addition in the bordism groups is induced by the disjoint union of these G-objects.

By the results of Section 4 we get G-cohomology theories connected by the structure of an equivariant cohomology theory that represent these bordism groups.

5.5 Symmetric Poincaré ad theories

The last two examples are the symmetric and quadratic Poincaré ad theories. Symmetric Poincaré ad theories are defined in Chapter 8 of [LM09]. Let R be a fixed ring with involution.

Definition 5.5.1. Let $\mathcal{C}$ denote the category of chain complexes of free left R modules. A chain complex C of $\mathcal{C}$ is called *finitely generated* if it is finitely generated in each degree and zero in all but finitely many degrees. It is *homotopy finite* if it is chain homotopy equivalent to a finitely generated object. The full subcategory of homotopy finite objects will be denoted by $\mathcal{D}$.

By applying the involution of R, one gets a chain complex of right R modules from an object C of $\mathcal{C}$, which will be denoted by C^t. Then $C^t \otimes_R C$ is equipped with the $\mathbb{Z}/2$ action that switches the factors. Let W be the standard resolution of $\mathbb{Z}$ by $\mathbb{Z}[\mathbb{Z}/2]$-modules (see [Wei94, ch. 6.2]).

Definition 5.5.2. A *quasi-symmetric complex of dimension* n is a pair (C, ϕ), where C is an object of $\mathcal{D}$ and ϕ is a $\mathbb{Z}/2$-equivariant map

$$W \to C^t \otimes_R C$$

of graded abelian groups which raises degrees by n.

Definition 5.5.3. Let $\mathcal{A}^R$ denote the category whose objects are the quasi-symmetric complexes and whose dimension-increasing morphisms $(C,\phi) \to (C',\phi')$ are the R-linear chain maps $f\colon C \to C'$. If the dimensions are equal, then one additionally requires that $(f^t \otimes f) \circ \phi = \phi'$ and there are no morphisms that lower dimension. Equip $\mathcal{A}^R$ with the involution which takes (C,ϕ) to $(C,-\phi)$, and let $\emptyset_n$ be the object for which C is zero in all degrees. Then $\mathcal{A}^R$ is a balanced $\mathbb{Z}$-graded category.

A balanced pre K-ad F with values in $\mathcal{A}^R$ will be denoted by

$$F(\sigma,o) = (C_\sigma, \phi_{\sigma,o}).$$

Again one wants F to fulfill certain properties: C should be well-behaved as a functor from $\mathcal{C}\mathrm{ell}^\flat(K)$ to chain complexes (see [LM09, Definition 8.7]) and F should be closed (see Definition 8.8). One denotes by $C_{\partial\sigma}$ the colimit $\operatorname{colim}_{\tau \subsetneq \sigma} C_\tau$. Then these properties ensure, that the composition

$$W \to C_\sigma^t \otimes_R C_\sigma \to (C_\sigma/C_{\partial\sigma})^t \otimes_R C_\sigma$$

is a chain map. Then for an oriented cell (σ,o) one chooses a right inverse $g\colon \mathbb{Z} \to W$ of the augmentation and defines a chain map

$$\Upsilon_\sigma\colon \operatorname{Hom}_R(C_\sigma, R) \to C_\sigma/C_{\partial\sigma}$$

to be the composition

$$\begin{aligned} \mathbb{Z} \otimes \operatorname{Hom}_R(C_\sigma,R) \to W \otimes \operatorname{Hom}_R(C_\sigma,R) &\to ((C_\sigma/C_{\partial\sigma})^t \otimes_R C_\sigma) \otimes \operatorname{Hom}_R(C_\sigma,R) \\ &\to (C_\sigma/C_{\partial\sigma})^t \otimes_R R \cong (C_\sigma/C_{\partial\sigma})^t. \end{aligned}$$

The chain homotopy class of this map is independent of the choice of g. Then one can define ads:

Definition 5.5.4. A pre K-ad F with values in $\mathcal{A}^R$ is a K-ad if it is balanced and closed and C is well-behaved and for each σ the map Υ_σ induces an isomorphism

$$H^*(\operatorname{Hom}_R(C_\sigma,R)) \to H_{\dim\sigma - \deg F - *}(C_\sigma/C_{\partial\sigma}).$$

The set of K-ads with values in $\mathcal{A}^R$ is denoted by $\mathrm{ad}^R(K)$.

Theorem 5.5.5 ([LM09, Theorem (8.13)(i)])**.** ad^R *is an ad theory.* □

These ad theories are called *symmetric Poincaré ad theories.*

Assuming again that Set and Ab are replaced by equivalent strict monoidal categories and additionally that R is commutative, $\mathcal{A}^R$ has a strict monoidal structure and the ad theory ad^R is multiplicative.

Proposition 5.5.6. *The symmetric Poincaré ad theories are ad theories with functorial gluing and cylinder constructions.*

Proof. Let K' be a subdivision of K. For a cell σ of K and a K'-ad F, gluing is constructed by taking the colimit D_σ of the underlying chain complexes of the values $F(\tau, o)$ with $\tau \in K'$ and $\tau \in \sigma$. The $\mathbb{Z}/2$-equivariant map $\kappa_{\sigma,o} \colon W \to D_\sigma^t \otimes_R D_\sigma$ is the sum of the compositions of the $\mathbb{Z}/2$-equivariant maps with the map into the tensor product $D_\sigma^t \otimes_R D_\sigma$ of this colimit and its associated complex of right modules, taken over all oriented cells (τ, o') of K' with $\dim \tau = \dim \sigma$ and $\tau \subset \sigma$ such that o' is induced by o. If F_1 and F_2 are two K'-ads of degree k, then we denote their values by $F_1(\tau, o') = (C_\tau^1, \phi_{\tau,o'}^1)$ and $F_2(\tau, o') = (C_\tau^2, \phi_{\tau,o'}^2)$ respectively. We denote the gluing constructions by $G(F_1)(\sigma, o) = (D_\sigma^1, \kappa_{\sigma,o}^1)$ and $G(F_2)(\sigma, o) = (D_\sigma^2, \kappa_{\sigma,o}^2)$. A natural transformation $g \colon F_1 \to F_2$ consists of R-linear chain maps $g_{(\tau,o')} \colon F_1(\tau, o') \to F_2(\tau, o')$ that fulfill $g_{(\tau,o')}^t \otimes g_{(\tau,o')} \circ \phi_{\tau,o'}^1 = \phi_{\tau,o'}^2$. So they induce a map $\widetilde{g}_{(\sigma,o)} \colon D_\sigma^1 \to D_\sigma^2$ between the colimits such that $\widetilde{g}_{(\sigma,o)}^t \otimes \widetilde{g}_{(\sigma,o)} \circ \kappa_{\sigma,o}^1 = \kappa_{\sigma,o}^2$, because

$$
\begin{aligned}
\widetilde{g}_{(\sigma,o)}^t \otimes \widetilde{g}_{(\sigma,o)} \circ \kappa_{\sigma,o}^1 &= \widetilde{g}_{(\sigma,o)}^t \otimes \widetilde{g}_{(\sigma,o)} \circ \sum_{(\tau,o')} (W \xrightarrow{\phi_{\tau,o'}^1} C_\tau^{1,t} \otimes C_\tau^1 \to D_\sigma^{1,t} \otimes D_\sigma^1) \\
&= \sum_{(\tau,o')} (\widetilde{g}_{(\sigma,o)}^t \otimes \widetilde{g}_{(\sigma,o)} \circ (W \xrightarrow{\phi_{\tau,o'}^1} C_\tau^{1,t} \otimes C_\tau^1 \to D_\sigma^{1,t} \otimes D_\sigma^1)) \\
&= \sum_{(\tau,o')} (W \xrightarrow{g_{(\tau,o')}^t \otimes g_{(\tau,o')} \circ \phi_{\tau,o'}^1} C_\tau^{2,t} \otimes C_\tau^2 \to D_\sigma^{2,t} \otimes D_\sigma^2)) \\
&= \sum_{(\tau,o')} (W \xrightarrow{\phi_{\tau,o'}^2} C_\tau^{2,t} \otimes C_\tau^2 \to D_\sigma^{2,t} \otimes D_\sigma^2)) \\
&= \kappa_{\sigma,o}^2.
\end{aligned}
$$

Hence the maps $\widetilde{g}_{(\sigma,o)}$ are morphisms of $\mathcal{A}^R$. The functoriality of taking colimit shows that $G(g)_{(\sigma,o)} := \widetilde{g}_{(\sigma,o)}$ defines a natural transformation and that G is a functor with this definition. This functor restricts to the identity functor on residual subcomplexes by definition.

The proof of the functoriality of the cylinder constructions is analogous to that for geometric Poincaré ad theories. Let K be a fixed ball complex and F a K-ad of degree k. Lemma (8.14) in [LM09] constructs tensor product ads analogously to the product ads of Lemma (7.11) and an I-ad G is constructed in the proof of Theorem (8.13) such that $J(F) = F \otimes G$. Hence J defines a functor $\mathrm{ad}^{R,k}(K) \to \mathrm{ad}^{R,k}(K \times I)$ and this functor restricts to the identity functor on $K \times 0$ and $K \times 1$ and it takes trivial ads to trivial ads. $\square$

Thus for every group G we get an ad theory of G-objects in $\mathcal{A}^R$ and associated G-cohomology theories that are connected by the structure of an equivariant cohomology theory.

5.6 Quadratic Poincaré ad theories

At last we present the quadratic Poincaré ad theories of Section 9 of [LM09].

Definition 5.6.1. A *quasi-quadratic complex of dimension* n is a pair (C, ψ) where C is an object of $\mathcal{D}$ and ψ is an element of $(W \otimes_{\mathbb{Z}/2} (C^t \otimes_R C))_n$.

Definition 5.6.2. Let $\mathcal{A}_R$ denote the category whose objects are the quasi-quadratic complexes and whose dimension-increasing morphisms $(C, \psi) \to (C', \psi')$ are the R-linear chain maps $f \colon C \to C'$. If dimensions are equal one additionally requires that $(1 \otimes (f^t \otimes f))\psi = \psi'$ and there are no morphisms that lower dimension. Equip $\mathcal{A}_R$ with the involution that takes (C, ψ) to $(C, -\psi)$ and $\emptyset_n$ is the n-dimensional object for which C is zero in all degrees. Then $\mathcal{A}_R$ is a balanced $\mathbb{Z}$-graded category.

A balanced pre K-ad F is denoted by

$$F(\sigma, o) = (C_\sigma, \psi_{\sigma,o}).$$

Again there is a definition of a property called *closed* (see Definition 9.3) one wants F to have. One defines a non-positively graded complex $V_0 \to V_{-1} \to \cdots$ of $\mathbb{Z}/2$-modules by

$$V_{-n} = \mathrm{Hom}_{\mathbb{Z}/2}(W_n, \mathbb{Z}[\mathbb{Z}/2]).$$

One gets an isomorphism

$$W \otimes_{\mathbb{Z}/2} (C^t \otimes_R C) \cong \mathrm{Hom}_{\mathbb{Z}/2}(V, C^t \otimes_R C),$$

so that the composition

$$N \colon W \to \mathbb{Z} \to V$$

induces a homomorphism

$$N^* \colon W \otimes_{\mathbb{Z}/2} (C^t \otimes_R C) \to \mathrm{Hom}_{\mathbb{Z}/2}(W, C^t \otimes_R C),$$

which is called the *norm map*. This map induces a functor $\mathcal{N} \colon \mathcal{A}_R \to \mathcal{A}^R$ by taking (C, ψ) to $(C, N^*(\psi))$.

Definition 5.6.3. A pre K-ad with values in $\mathcal{A}_R$ is a K-ad if it is balanced, closed and C is well-behaved and $\mathcal{N} \circ F$ is a K-ad.

Remark 5.6.4. Note that here well-behaved means the same as for symmetric Poincaré ads.

Theorem 5.6.5 ([LM09, Theorem (9.5)])**.** ad_R *is an ad theory.* □

These ad theories are called *quadratic Poincaré ad theories.*

Proposition 5.6.6. *The quadratic Poincaré ad theories are ad theories with functorial gluing and cylinder constructions.*

Proof. This is proved in the same way as for symmetric Poincaré ad theories with the only difference, that now Lemma (9.6) of [LM09] plays the role of Lemma (8.14). □

Again for every group G we get an ad theory of G-objects in $\mathcal{A}_R$ and associated G-cohomology theories that are connected by the structure of an equivariant cohomology theory.

5.7 Outlook on other equivariant constructions concerning symmetric and quadratic Poincaré ad theories

An R-module with a G-action is the same as an RG-module. This is not true for free R-modules and free RG-modules and because of this we can not understand the categories of G-objects in $\mathcal{A}^R$ and $\mathcal{A}_R$ by the categories $\mathcal{A}^{RG}$ and $\mathcal{A}_{RG}$.

However it seems natural that one wants to get equivariant theories by the dependency on G of the symmetric and quadratic ad theories associated to RG for a fixed ring with involution R. In fact in [DL98] a covariant functor from GROUPOIDS to L-theory spectra is given that takes equivalences of groupoids to weak equivalences of spectra:

For a small groupoid $\mathcal{C}$ a so-called *R-category with involution $R\mathcal{C}$* is constructed. Furthermore one can define a *symmetric monoidal R-category $R\mathcal{C}_{\oplus}$*. If $\mathcal{C}$ is the category of a group, then $R\mathcal{C}$ is simply the group ring.

Now one can apply the usual construction of the periodic algebraic L-theory spectrum to these R-categories and gets the desired covariant functors GROUPOIDS $\rightarrow \mathcal{S}$, that take equivalences of groupoids to weak equivalences of spectra.

An advantage of this construction is that assembly maps and the isomorphism conjecture of Farrell-Jones can then be formulated in terms of or(G)-spectra.

The hope is, that such a construction is also possible for symmetric and quadratic Poincaré ad theories. Gerd Laures and James E. McClure provide a (covariant) functoriality in R of symmetric and quadratic ad theories in Section 11 of [LM09] (they use a method of Blumberg and Mandell and the target categories have to be modified). A part of the strategy would be to use this functoriality.

The advantage would be, that for example with symmetric Poincaré ad theories and if R is commutative, one would get functors from GROUPOIDS to symmetric ring spectra.

I intend to pursue this idea in my future work.

6 Appendix

6.1 Some facts from point-set topology

Throughout this work all topological spaces are assumed to be compactly generated weak Hausdorff spaces and we mean the category of this spaces when we speak of topological spaces. We write Top for this category itself, $\mathrm{Top}(X,Y)$ for the set of continuous maps $X \to Y$ and $\mathrm{Map}(X,Y)$ for the mapping space in this category, which is equipped with the compactly generated compact-open topology. Working with this category ensures that it fulfills an *exponential law*, that is for spaces X, Y and Z we get a homeomorphism

$$\mathrm{Map}(X \times Y, Z) \cong \mathrm{Map}(X, \mathrm{Map}(Y,Z))$$

and these homeomorphisms are natural in X, Y and Z. See [tD00, VI.6] or [Whi78, I.4] for details.

In particular if X and Y are locally compact Hausdorff spaces, then $\mathrm{Map}(X,Y)$ has the compact-open topology.

Lemma 6.1.1. *Let $\mathcal{I}$ be a finite category and F be a functor from $\mathcal{I}$ to locally compact Hausdorff spaces such that $\mathrm{colim}_{\mathcal{I}} F$ is a locally compact Hausdorff space. Let Y be a locally compact Hausdorff space. Then the bijection induced by the universal property of the colimit*

$$\mathrm{Map}(\operatorname*{colim}_{\mathcal{I}} F, Y) \to \lim_{\mathcal{I}} \mathrm{Map}(F(-), Y)$$

is a homeomorphism.

Proof. The map from the left to the right side is continuous by the universal property of the limit. For finite disjoint sums one can check the continuity of the inverse map directly using the definition of the compact-open topology. If $X \to Z$ is an identification map (that is Z carries the quotient topology) between locally compact Hausdorff spaces, then we have to prove that $\mathrm{Map}(Z,Y)$ has the induced topology of the inclusion

$$\mathrm{Map}(Z,Y) \to \mathrm{Map}(X,Y).$$

That is we have to check that for each test space T a map $f\colon T \to \mathrm{Map}(Z,Y)$ is continuous if and only if the composition of the inclusion with f is continuous. By the exponential law this is equivalent to $T \times X \to T \times Z$ being an identification. This is true by theorems (6.8) and (6.13) of [tD00].

We have shown that the claim holds for finite disjoint sums and quotient spaces, hence it is true for all finite colimits. □

Proposition 6.1.2. *Let $\mathcal{I}$ be a finite category and F_1 and F_2 functors from $\mathcal{I}$ to locally compact Hausdorff spaces, such that* $\mathrm{colim}\, F_1$ *and* $\mathrm{colim}\, F_2$ *are locally compact Hausdorff spaces. Let T be a topological space and let there be continuous maps*

$$f_J\colon T \to \mathrm{Map}(F_1(J), F_2(J))$$

for every object J of $\mathcal{I}$, which are compatible, that is for morphisms in $\mathcal{I}$ the induced diagrams commute. Then the colimit map

$$T \quad \to \quad \mathrm{Map}(\operatorname*{colim}_{\mathcal{I}} F_1, \operatorname*{colim}_{\mathcal{I}} F_2)$$

is continuous.

Proof. Let in_J denote the continuous map $F_2(J) \to \operatorname{colim}_{\mathcal{I}} F_2$. Then the composition

$$T \to \mathrm{Map}(F_1(J), F_2(J)) \to \mathrm{Map}(F_1(J), \operatorname*{colim}_{\mathcal{I}} F_2),$$

where the second map is given by composition with in_J, is continuous. By the universal property of the limit we get a unique continuous map

$$T \to \lim \mathrm{Map}(F_1(J), \operatorname*{colim}_{\mathcal{I}} F_2).$$

Now we can apply Lemma 6.1.1 to the right side and simply check that the resulting map is the colimit map. □

Lemma 6.1.3. *Let X, Y and Z be locally compact Hausdorff spaces. Then the bijection*

$$\mathrm{Map}(X, Y) \times \mathrm{Map}(X, Z) \to \mathrm{Map}(X, Y \times Z)$$

given by the universal property of the product is a homeomorphism.

Proof. The map from the right to left side is continuous by the universal property of the product. The continuity of the other direction can be checked directly using the definitions of the product topology and the compact-open topology. □

Proposition 6.1.4. *Let Y_1, Y_2 and W be locally compact Hausdorff spaces, T a topological space and f a continuous map*

$$T \to \mathrm{Map}(Y_1, Y_2).$$

Then the map induced by the product with W

$$T \to \mathrm{Map}(Y_1 \times W, Y_2 \times W)$$

is continuous.

Proof. We compose f with the continuous map induced by the composition with the projection $Y_1 \times W \to Y_1$ and get a continuous map $T \to \mathrm{Map}(Y_1 \times W, Y_2)$. On the other hand we can compose the constant map $T \to \mathrm{Map}(W, W)$ with the map induced by composition with the projection $Y_1 \times W \to W$ and get a continuous map $T \to \mathrm{Map}(Y_1 \times W, W)$. Together these maps define a continuous map

$$T \to \mathrm{Map}(Y_1 \times W, Y_2) \times \mathrm{Map}(Y_1 \times W, W)$$

to whose right side we can apply Lemma 6.1.3. Then one sees that the resulting continuous map is the map whose continuity we wanted to show. □

6.2 Ball complexes

We will not give an extensive introduction to piecewise linear topology here. Instead we refer to [RS72] where the foundations are elaborated in every detail. Ball complexes are described in [BRS76, I.1] and the basic definitions are also given in [LM09]. We will shortly recall the definitions concerning ball complexes which are the technical device we need for the definition of ad theories. After that we will explain what is meant by the incidence number of two cells which occurs in the definition of ad theories.

An m-dimensional *p.l. manifold* is a polyhedron M, so that every point $x \in M$ has a neighborhood which is p.l. homeomorphic to an open subset of $\mathbb{R}^n$. An m-dimensional *p.l. ball* is a p.l. manifold with boundary which is p.l. homeomorphic to the p.l. manifold I^m.

Definition 6.2.1. For a finite collection K of p.l. balls in some $\mathbb{R}^n$ we define $|K|$ to be the union $\bigcup_{\sigma \in K} \sigma$. Then K is called a *ball complex* if the interiors of the balls are disjoint and for each ball $\sigma \in K$ its boundary is a union of balls of K.

The balls of a ball complex are often called *closed cells* or only *cells*. A ball complex is a regular CW complex (see [Whi78, II.6] for a definition of regular CW complexes). We sometimes call $|K|$ the *underlying space* of K. Every finite simplicial complex is a ball complex.

Definition 6.2.2. A subset of a ball complex K which is itself a ball complex is called a *subcomplex*. An isomorphism of ball complexes is a p.l. homeomorphism which takes cells to cells. If L is a subcomplex of K then we denote the *pair of ball complexes* by (K, L). An isomorphism of pairs (K, L) and (K', L') is an isomorphism of ball complexes $K \to K'$ which restricts to an isomorphism $L \to L'$. A *morphism* f of ball complexes is a composition of an isomorphism ϕ with an inclusion j of a subcomplex: $f = j \circ \phi$. Similarly, a morphism of pairs is a composition of an isomorphism of pairs with an inclusion of pairs. Let Bi be the category of pairs of ball complexes. The category with the same objects whose morphisms are homotopy classes of continuous maps of pairs will be denoted by Bh.

The product of ball complexes is defined by the products of the cells and it is again a ball complex (see [BRS76, page 5]). This is an advantage of ball complexes over simplicial complexes. We will often use products with the ball complex I which is the unit interval together with its standard structure as a ball complex, which consists of two 0 cells and one 1 cell.

Definition 6.2.3. Let K be a ball complex. A ball complex K' is called a *subdivision* of K if it fulfills the two conditions

(i) $|K'| = |K|$

(ii) Each cell of K' is contained in a cell of K.

Let σ be an n-dimensional cell and write $\partial\sigma$ for its boundary. Then there is an isomorphism $H_n(\sigma, \partial\sigma) \cong \widetilde{H}_n(S^n) \cong \mathbb{Z}$ where H denotes singular homology and $\widetilde{H}$ denotes reduced homology. We define an orientation of the cell to be a generator o of $H_n(\sigma, \partial\sigma)$. We write (σ, o) for the oriented cell.

Now let (σ, o) be a cell of dimension n of a ball complex K with chosen orientation o and let (τ, o') a cell of dimension $n-1$ of K with chosen orientation o'. Let τ be contained in the boundary of σ. Then we have the *incidence isomorphism* as is described in [Whi78, p. 82]:

$$H_n(\sigma, \partial\sigma) \cong H_{n-1}(\tau, \partial\tau).$$

The *incidence number* $[(\sigma, o), (\tau, o')]$ is defined to be 1 if o is thrown to o' under this isomorphism and -1 if o is thrown to $-o'$. If τ is not contained in the boundary of σ the incidence number is defined to be 0.

6.3 Foundations of equivariant topology

We collect basic definitions and properties of equivariant topology here. In particular, we will introduce the orbit category. Of course we will not give proofs for every detail here, instead we refer the reader to the books [tD87] and [May96].

Let G be a group. The category of the group G is the category which consists of one object $*$ with the set of endomorphisms of $*$ being the group G and composition is the multiplication of the group. Thus all endomorphisms are automorphisms and e is the identity. We denote this category by $\mathcal{G}$.

Now let C be an object of a category $\mathcal{C}$. There are several (in their specific situation equivalent) notions of a G-action on C which are:

- A *(left-)action* of G on C is a group homomorphism $G \to \mathrm{Aut}(C)$ to the group of automorphisms of C.
- A (left-)action of G on C is a functor $F\colon \mathcal{G} \to \mathcal{C}$ such that $F(*) = C$.
- If $\mathcal{C}$ is the category of sets, then a (left-)action of G on a set X is a map

  $$G \times X \to X, \quad (g, x) \mapsto gx$$

  such that $(gg')x = g(g'x)$ for all g, $g' \in G$ and $ex = x$ where e denotes the neutral element of G. Sometimes one writes $g.x$ for gx, for example if one wants to distinguish between an action and a group multiplication. By the exponential law this is equivalent to the first definition.

We omit the term *left* in the following, and note that there is also a notion of right action: A *right-action* of G on some object C of $\mathcal{C}$ is a contravariant functor $F\colon \mathcal{G} \to \mathcal{C}$ or if $\mathcal{C}$ is the category of sets and X a set, then this is the same as a map

$$X \times G \to X, \quad (x, g) \mapsto xg$$

such that $xe = x$ and $(xg)g' = x(gg')$. Note that left actions can be transformed to right actions and vice versa by the formula $xg = g^{-1}x$.

An object of $\mathcal{C}$ with a group action of G on it is called a *G-object* in $\mathcal{C}$.

Often we are in the situation of a category $\mathcal{C}$ whose objects are sets with additional structures. For example, think of topological spaces or manifolds or (left) modules over a ring R. Then an action of G on C gives an action of G on the set C by applying the forgetful functor from $\mathcal{C}$ to the category of sets to the action. The resulting action on the set has the special property that it *acts by automorphisms* in the category $\mathcal{C}$, that is for all elements $g \in G$ the map of sets induced by it $C \to C$, $c \mapsto gc$ is an automorphism in the category $\mathcal{C}$. On the other hand every G-action on the underlying set of C by automorphisms of $\mathcal{C}$ is a G-object in $\mathcal{C}$.

Now let G be a topological group, that is G is a group together with a topology such that the multiplication map and the inverse map

$$\begin{aligned} G \times G \to G, &\quad (g, g') \mapsto gg' \\ G \to G, &\quad g \mapsto g^{-1} \end{aligned}$$

are both continuous.

Definition 6.3.1. A *left-G-space* is a topological space X together with a continuous map

$$G \times X \to X, \quad (g, x) \mapsto gx$$

which fulfills $ex = x$ and $g'(gx) = (gg')x$. Such a map is called a *continuous left-action* of G on X.

Again we will often omit the term *left*, when speaking of G-spaces, and sometimes even *continuous* when it is clear that we talk about a continuous action. There is a notion of continuous right actions as for G-actions.

Note that a G-space X is more special than a G-object in Top because it describes continuous symmetries. Every G-space is a G-object in Top. Conversely, this is not true in general. The categories $\mathcal{G}$ and Top can be viewed as topological categories, in the sense that their morphism sets carry a topology such that composition is continuous. The topology of the (only) morphism set of $\mathcal{G}$ is that of G and the topology on the morphism sets of Top is given by the mapping spaces. Then a G-space is a continuous functor in the sense, that the maps on the morphism sets are continuous. This is essentially the same as a continuous group homomorphism $G \to \mathrm{Aut}(X)$, where $\mathrm{Aut}(X)$ is the group of homeomorphisms $X \to X$ with the subspace topology of the mapping space topology.

If G has the discrete topology, then a G-space is the same as a G-object in Top.

We call an action *effective* if its group homomorphism is injective.

Definition 6.3.2. A *Lie Group* is a group G which is a smooth manifold such that the multiplication map and inverse map are both smooth maps.

Usually one regards compact Lie groups in equivariant topology. Often one even restricts oneself to the case of finite groups. When we speak about a finite group G we always assume that it is equipped with the discrete topology. In particular every G-action on a space is continuous and every subgroup is closed.

A continuous map between G-spaces $f : X \to Y$ is called *G-equivariant* if $f(gx) = gf(x)$ for all $g \in G$ and $x \in X$. Such a map is also called a *G-map*. The G-spaces and G-maps form a category which we denote by Top_G. Generally constructions for spaces like sums and products work well for G-spaces. For instance, we get Cartesian products by letting G act diagonally. For G-spaces X and Y the group G acts on $\mathrm{Map}(X, Y)$ by conjugation, that is $gf(x) := gf(g^{-1}x)$. With this action the exponential law provides a G-homeomorphism

$$\mathrm{Map}(X \times Y, Z) \cong \mathrm{Map}(X, \mathrm{Map}(Y, Z))$$

Additionally we define $\mathrm{Map}_G(X, Y)$ to be the subspace of $\mathrm{Map}(X, Y)$ which contains the G-maps.

For a G-action on X the *orbit* of x is the set $Gx := \{gx \mid g \in G\}$. The orbits provide a disjoint partition of X, so they define an equivalence relation. The quotient space is denoted by X/G and is called the *orbit space* of the action.

Next we regard subgroups. For us all subgroups that appear are assumed to be topologically closed. Note that this is always true for subgroups of a finite group. Let $H \subset G$ be a subgroup. We have a continuous right action of H on G by right translation. The orbits of this action are the cosets gH and the orbit space G/H is a left-G-space by the action

$$G \times G/H \to G/H, \quad (g', gH) \mapsto g'gH$$

A G-space is called *homogeneous space* if it is G-homeomorphic to G/H for some subgroup $H \subset G$. The G-spaces G/H together with G-maps between them form a category which is called the *orbit category* of G and is denoted by $\mathrm{or}(G)$.

Let X^H denote the *fixed point space* of X under the action of H, that is

$$X^H := \{x \mid hx = x \text{ for } h \in H\}$$

equipped with the subspace topology. For an element $x \in X$ the group

$$G_x := \{g \mid gx = x\}$$

is called the *isotropy group* of x.

The *normalizer* of a subset H of a group G is defined by $N_G H := \{g \mid gHg^{-1} = H\}$. Therefore a subgroup H is normal in G if and only if $N_G H = G$. For a subgroup the normalizer is again a group, which contains H as a normal subgroup. Indeed it is the largest subgroup of G with this property. The quotient group $WH := N_G H/H$ is called the *Weyl group* of H.

Lemma 6.3.3. *X^H is a WH-space.*

Proof. The action is defined by $[g]x := gx$. We have to show that this is well-defined. Let g be a representative for $[g]$. First, $gx \in X^H$ because $hgx = gh'x = g.x$ for some $h' \in H$ which exists by the definition of the normalizer. For $[g'] = [g] = gH$ we know that $g' = gh$ for some $h \in H$. So $g'x = ghx = gx$. The action is continuous because the action of $N_G H$ on X^H is, and WH is equipped with the quotient topology. □

We cite a simple but useful proposition from [tD87]:

Proposition 6.3.4 ([tD87, (I.1.14)]). *Let H and K be subgroups of G.*

(i) There exists a G-map $G/H \to G/K$ if and only if H is conjugate to a subgroup of K.

(ii) If $a \in G$, $a^{-1}Ha \subset K$, then we obtain a G-map $R_a\colon G/H \to G/K$, $gH \mapsto gaK$.

(iii) Each G-map $G/H \to G/K$ has the form R_a for suitable $a \in G$ with $a^{-1}Ha \subset K$.

(iv) $R_a = R_b$ if and only if $ab^{-1} \in K$. □

For a G-space X let $\mathrm{Aut}_G(X)$ denote the set of G-homeomorphisms $X \to X$.

Corollary 6.3.5. *For a subgroup $H \subset G$ the group $\mathrm{Aut}_G(G/H)$ is isomorphic to the Weyl group WH.*

Proof. See page 6 of[tD87]. □

Corollary 6.3.6. *We have bijections*

$$\mathrm{Top}_G(G/H, G/K) \cong (G/K)^H$$

Proof. By the proposition the set $\mathrm{Top}_G(G/H, G/K)$ can be identified with the set of equivalence classes $[a]$ of elements $a \in G$ with $a^{-1}Ha \subset K$ where the equivalence relation is given by $a \sim b :\Leftrightarrow a^{-1}b \in K$. On the other hand the elements of $(G/K)^H$ are cosets aK with $haK = aK$ for all $h \in H$. Now it is easy to see that there are well-defined maps $[a] \mapsto aK$ and $aK \mapsto [a]$, which are inverse to each other by definition. □

Lemma 6.3.7. *For a G-space X and a subgroup $H \subset G$ the orbit space X/H is a WH-space.*

Proof. The action is defined by $[g]Hx := H(gx)$. This is well-defined, because if $[g] = [g']$, then $g' = gh$ for some $h \in H$ and because of $g \in NH$ we have $gh = h'g$ for some $h' \in H$ and so $H(g'x) = H(h'gx) = H(gx)$. □

We continue by collecting several adjunctions we may need. Let K be a G-space with trivial G-action. The proof of the following two bijections is simple.

Proposition 6.3.8. *We have bijections*

$$\mathrm{Top}_G(K, X) \cong \mathrm{Top}(K, X^G)$$

and

$$\mathrm{Top}_G(X, K) \cong \mathrm{Top}(X/G, K).$$

Definition 6.3.9. Let G and H be groups and X be an H-space and $\alpha\colon H \to G$ a group homomorphism. On the product $G \times X$ an equivalence relation is defined by $(g\alpha(h), x) \sim (g, hx)$. Let $\operatorname{ind}_\alpha X$ denote the quotient space of this relation. It is the same as the quotient of $G \times X$ of the right action given by $(g,x)h := (g\alpha(h), h^{-1}x)$. It is equipped with a G-action

$$G \times (G \times_H X) \to G \times_H Y, \quad (g, [(g', x)]) \mapsto [(gg', x)]$$

and we call this G-space the *induction of X with α*. If α is an inclusion of a subgroup, we also write $\operatorname{ind}_H^G X$ or $G \times_H X$ for it.

Now let $\alpha\colon H \to G$ be the inclusion of a subgroup. Then the construction of the induced G-space defines a functor $\mathrm{Top}_H \to \mathrm{Top}_G$. We have a functor in the other direction by restricting the G-action to H. These functors are adjoint:

Proposition 6.3.10. *Let X be an H-space and Y a G-space. There is a canonical bijection*

$$\mathrm{Top}_G(G \times_H X, Y) \cong \mathrm{Top}_H(X, Y)$$

Proof. See proposition (I.4.3) in [tD87]. □

Generally for a right G-space X and a left G-space Y we define the *balanced product* $X \times_G Y := (X \times Y)/\sim$ with $(xg, y) \sim (x, gy)$. If both X and Y are left-G-spaces then we regard X with the associated right action to get the balanced product. For an H-space X this definition agrees with that for $G \times_H X$.

Dually to the induced G-space we can construct a *coinduced* G-space. Let $\mathrm{Map}_H(G, X)$ be the space of H-maps $G \to X$ where H acts on G by multiplication from the left. It is a G-space by the action $gf(g') = f(g'g)$. Then this G-space is called the coinduced G-space. We get the dual version of the canonical bijection:

Proposition 6.3.11. *Let X be an H-space and Y a G-space. There is a canonical bijection*

$$\mathrm{Top}_H(Y, X) \cong \mathrm{Top}_G(Y, \mathrm{Map}_H(G, X))$$

Proof. See [tD87, I.4.11]. □

The following statement provides alternative descriptions of the induced and coinduced G-space.

Proposition 6.3.12. *Let X be a G-space. Then we have G-homeomorphisms*

$$G \times_H X \cong (G/H) \times X$$

and

$$\mathrm{Map}_H(G, X) \cong \mathrm{Map}(G/H, X)$$

Proof. The first G-homeomorphism is given by the map $[(g, x)] \mapsto (gH, gx)$. This map is a well-defined G-map and its inverse is given by the G-map $(gH, x) \mapsto [(g, g^{-1}x)]$. For the second G-homeomorphism let $f\colon G/H \to X$ be a continuous map. Then define $f' : G \to X$ by $f'(g) = gf(g^{-1}H)$. This is an H-map because $f'(hg) = hgf(g^{-1}h^{-1}H) = hgf(g^{-1}H) = hf'(g)$. For the other direction let $f\colon G \to X$ be an H-map. Define $f'(gH) = gf(g^{-1})$. It is easy to see that these constructions are inverse to each other. □

We go on by explaining homotopies in the equivariant context. Let I be equipped with the trivial G-action. Then for every G-space X the product $X \times I$ is a G-space. Two G-maps $X \to Y$ are defined to be *G-homotopic* if there is a G-map $h\colon X \times I \to Y$ which restricts to the two G-maps on the ends. G-homotopy is an equivalence relation and we denote the homotopy category by hTop_G.

A G-map $f\colon X \to Y$ is called a *weak equivalence* if the restrictions $f^H\colon X^H \to Y^H$ are weak equivalences for all subgroups $H \subset G$. If we add formal inverses for weak equivalences to hTop_G we denote the resulting category by $\bar{\mathrm{h}}\mathrm{Top}_G$.

In homotopy theory one often works with based spaces. A *based G-space* is a G-space together with a point of it which is G-fixed. Let X_+ denote the disjoint union of a G-space X and a base point on which G acts trivially. We have to replace the Cartesian product by the smash product $X \wedge Y = (X \times Y)/X \vee Y$ to get a suitable product in the category of based G-spaces. The morphisms are based G-maps and we denote the set of morphisms by $\mathrm{Top}_G^{\mathrm{pt}}(X, Y)$ or by $\mathrm{Top}_G(X, Y)$ if it is clear that we work in the category of based G-spaces.

We get analogous adjunctions and bijections for the based situation. A based homotopy between based G-maps $X \to Y$ is a based G-map $h\colon X \wedge I_+ \to Y$ which restricts to the maps on the ends. Again we get analogous homotopy categories.

We proceed by introducing the fixed point functor and the orbit functor. For the rest of this section let X be a G-space and G a compact topological group.

Lemma 6.3.13. *We get a contravariant functor from the orbit category* $\mathrm{or}(G)$ *to topological spaces as follows:*

$$G/H \mapsto X^H$$
$$(R_a\colon G/H \to G/K) \mapsto \begin{cases} X^K \to X^H \\ x \mapsto ax \end{cases}$$

Proof. We only have to verify that the functor is well-defined on morphisms. First we know that for a morphism R_a we have $a^{-1}Ha \subset K$, so for $x \in X^K$ we have $hax = akx$ for some $k \in K$ and therefore $hax = ax$. If $R_a = R_b$ then $ab^{-1} \in K$. So for $x \in X^K$ we get $ax = bx$. □

We call this functor the *fixed point functor*.

Proposition 6.3.14 ([tD87, I.3.8])**.** *For a compact group G there is a canonical homeomorphism*

$$\mathrm{Map}_G(G/H, X) \cong X^H$$

which is given by $f \mapsto f(H)$.

With this proposition the fixed point functor is isomorphic to the functor $\mathrm{Map}_G(-, X)$, because $(\mathrm{Map}_G(-, X)(R_a)(f))(H) = (f \circ R_a)(H) = f(aK) = af(K)$.

Lemma 6.3.15. *A covariant functor from the orbit category to topological spaces is given by*

$$G/H \mapsto X/H$$
$$(R_a \colon G/H \to G/K) \mapsto \begin{cases} X/H \to X/K \\ Hx \mapsto Kax \end{cases}$$

Proof. Again we only have to proof that it is well-defined on morphisms. For a morphism R_a we have $Ha = aK$. So if $Hx = Hy$ it follows that $y = hx$ for some $h \in H$ and $ah = ka$ for some $k \in K$. Thus $Kay = Kahx = Kkax = Kax$. If $R_a = R_b$ then $ab^{-1} \in K$, so $Kbx = Kab^{-1}bx = Kax$. Continuity follows from the continuity of the left translation and the universal property of quotient topology. □

This functor is called the *quotient space functor*.

Proposition 6.3.16. *Let X be a left G-space. There is a canonical homeomorphism*

$$X/H \cong X \times_G G/H.$$

Proof. The map is given by $Hx \mapsto [x, H]$, the inverse map is given by $[x, gH] \mapsto Hg^{-1}x$. It is easy to check that both maps are well-defined and that they are continuous. □

This proposition yields an isomorphism of the quotient space functor to the functor $X \times_G -$, because $X \times_G -(R_a)([x, H]) = [x, aK] = [ax, a^{-1}aK] = [ax, K]$.

Definition 6.3.17. Let G be a group. A *family of subgroups* $\mathcal{F}$ of G is a nonempty set of subgroups which is closed under taking subgroups and conjugation with elements of G. We denote by $\mathrm{or}(G, \mathcal{F})$ the full subcategory of $\mathrm{or}(G)$ consisting of the objects G/H with $H \in \mathcal{F}$.

Example 6.3.18. Let G be a group, then there are the following examples of families of subgroups:

- $\mathcal{F} = \{H \subset G \mid H \text{ is a finite subgroup of } G\}$
- If X is a G-space then $\mathcal{F} = \{H \subset G \mid X^H \neq \emptyset\}$ is a family of subgroups.
- A group is called *virtual cyclic* if it contains a cyclic subgroup of finite index. Let

  $$\mathcal{F} = \{H \subset G \mid H \text{ is virtual cyclic.}\}$$

  Then $\mathcal{F}$ is a family of subgroups.

We proceed by giving a short introduction to G-CW-complexes. See [May96, I.3] or [KL05, 19.1] or [tD87, II.1, II.2] for details. Let G be a topological group and let the disks D^n and the spheres S^n be equipped with the trivial G-action. Note that the homogeneous spaces G/H play the role of points in equivariant topology.

Definition 6.3.19. A G-CW complex X is a G-space together with a G-invariant filtration

$$\emptyset = X_{n-1} \subseteq X_0 \subseteq X_1 \subseteq \ldots \subseteq X_n \subseteq \ldots \bigcup_{n\geq 0} X_n = X$$

such that

(i) X carries the colimit topology of this filtration.

(ii) for all $n \geq 0$ the G-space X_n is obtained from X_{n-1} by attaching equivariant n-dimensional cells, that is there exists a G-pushout

$$\begin{tikzcd}
\coprod_{i\in I_n} G/H_i \times S^{n-1} \arrow[r] \arrow[d] & X_{n-1} \arrow[d] \\
\coprod_{i\in I_n} G/H_i \times D^n \arrow[r] & X_n
\end{tikzcd}$$

Note that the attaching maps $G/H \times S^n \to X_n$ are determined by their restrictions to the fixed point sets $S^n \to X_n^H$, because of the adjunctions above. This can be used to prove homotopy theoretical results as for non-equivariant CW complexes: There is a Whitehead theorem (see [May96, Theorem 3.2] or [KL05, Theorem 19.6]) and a cellular approximation theorem (see [May96, Theorem 3.6] or [tD87, Theorem II.2.1]). A G-map $f\colon X \to Y$ of G-CW complexes is *cellular* if $f(X_n) \subseteq Y_n$ for every n. Then the cellular approximation theorem ensures that each G-map between G-CW-complexes is G-homotopic to a cellular one and each G-homotopy between cellular G-maps can be replaced by a cellular G-homotopy.

6.4 Spectra and symmetric spectra

There exist several new model categories, which are all Quillen equivalent and whose associated homotopy categories are equivalent to the classical stable homotopy category. Some of them are constructed from diagram categories of spectra. The different constuctions of such model categories of diagram spectra are covered and compared in [MMSS01]. One of these categories is the category of (topological) symmetric spectra of [HSS00]. The advantage of all these new examples for model categories for the stable homotopy category is, that a reasonable smash product can already be defined on the category itself and not just on the homotopy category.

The main purpose of the machinery given in [LM09] is to take a multiplicative ad theory and produce symmetric ring spectra. In this section we collect basic definitions for spectra and symmetric ring spectra from the literature.

Definition 6.4.1. A *spectrum* X is a family $(X_n)_{n\in\mathbb{Z}}$ of pointed topological spaces together with maps

$$\epsilon_n \colon X_n \wedge S^1 \to X_{n+1}.$$

A spectrum is called Ω-*spectrum* if the adjoint maps $\sigma_n \colon X_n \to \Omega X_{n+1}$ are weak equivalences, that is they induce isomorphisms of the homotopy groups. A map of spectra $f\colon X \to Y$ is a family of maps $f_n \colon X_n \to Y_n$ such that for all $n \in \mathbb{Z}$ the diagrams

$$\begin{tikzcd} X_n \wedge S^1 \arrow[r, "\epsilon_n"] \arrow[d, "f_n\wedge \mathrm{id}_{S^1}"'] & X_{n+1} \arrow[d, "f_{n+1}"] \\ Y_n \wedge S^1 \arrow[r, "\epsilon_n"'] & Y_{n+1} \end{tikzcd}$$

commute. We write $\mathcal{S}$ for the category of spectra and Ω-$\mathcal{S}$ for the category of Ω-spectra.

Spectra represent generalized homology and cohomology theories by Brown's representability theorem. Let S^k denote the k-fold smash product of S^1. The symmetric group Σ_k (equipped with the discrete topology) acts continuously on S^k by permutations of the factors.

Definition 6.4.2. A *symmetric spectrum* X is a spectrum together with continuous actions of the symmetric group Σ_n on X_n such that for all $n, k \in \mathbb{Z}$ the composition

$$X_n \wedge S^k \xrightarrow{\epsilon_n \wedge \mathrm{id}_{S^{k-1}}} X_{n+1} \wedge S^{k-1} \longrightarrow \cdots \longrightarrow X_{n+k}$$

is $\Sigma_n \times \Sigma_k$-equivariant ($\Sigma_n \times \Sigma_k$ acts on X_{n+k} as subgroup of Σ^{n+k}). A *morphism of symmetric spectra* is a map of the spectra such that $f_n \colon X_n \to Y_n$ is Σ_n-equivariant. We denote the category of symmetric spectra by $\mathcal{S}^\Sigma$.

Definition 6.4.3. A *symmetric sequence* is a sequence of pointed spaces $X_0, X_1, \ldots$ together with actions of Σ_n on X_n. A morphism $X \to Y$ between symmetric sequences consists of Σ_n-equivariant maps $X_n \to Y_n$.

On the category of symmetric sequences there exists a tensor product $\otimes$ (see [HSS00, Definition 2.1.3]) and a twist isomorphism τ (see [HSS00, Proposition 2.1.4 and Remark 2.1.5]), such that this category is symmetric monoidal ([HSS00, Lemma 2.1.6]). If S is the symmetric sequence of the symmetric sphere spectrum ([HSS00, Example 1.2.4]), then there is an equivalence between left S-modules in the category of symmetric sequences and symmetric spectra ([HSS00, Proposition 2.2.1]).

Definition 6.4.4. Let X, Y be symmetric spectra and $m_X \colon S \otimes M \to M$ and $m_Y \colon S \otimes Y \to Y$ the structure maps of the left S-modules. Then

$$X \wedge Y := \text{coequalizer}(\ X \otimes S \otimes Y \underset{(m_X \circ \tau)\otimes \mathrm{id}_Y}{\overset{\mathrm{id}_X \otimes m_Y}{\rightrightarrows}} X \otimes Y\ )$$

defines a smash product in the category of symmetric spectra that makes $\mathcal{S}^\Sigma$ a symmetric monoidal category ([HSS00, Lemma 2.2.2 and Corollary 2.2.4]).

6.5 Diagrams in a category and evaluation functors

Often additional structures on objects of a category (for example topological spaces) can be described by functors into this category. For instance, recall that a G-object in a category is a functor of $\mathcal{G}$ into the category. Another example is given by or(G)-spaces which are defined to be functors from or(G) to spaces. For a G-space X a contravariant functor from or(G) to spaces is given by the fixed point functor $G/H \mapsto X^H$ and a covariant functor is defined by the quotient space functor.

One idea is that one can get a lot of information about the equivariant homotopy theory from the system of fixed point sets. In this section we want to introduce the basic terminology of diagrams in a category. For further treatment we refer to [DL98] and [DF87].

Let $\mathcal{C}$ be a small category and $\mathcal{D}$ a category.

Definition 6.5.1. A covariant (contravariant) functor $\mathcal{C} \to \mathcal{D}$ is called a covariant (contravariant) $\mathcal{C}$*-diagram* in $\mathcal{D}$.

Recall that a contravariant functor is a covariant functor from the dual category $\mathcal{C}^{op}$ to $\mathcal{D}$. Having that in mind, we omit the variance in what follows and give definitions only for the covariant case.

If $\mathcal{D}$ is the category of topological spaces, we speak of $\mathcal{C}$-diagrams of spaces, if $\mathcal{D}$ is the category of topological manifolds, then we speak of $\mathcal{C}$-diagrams of manifolds and so on. All $\mathcal{C}$-diagrams in $\mathcal{D}$ form a category whose morphisms are the natural transformations. We denote this functor category by $\mathcal{D}^{\mathcal{C}}$ or sometimes by $\mathcal{C}$-$\mathcal{D}$ and call it the *category of* $\mathcal{C}$*-diagrams in* $\mathcal{D}$ or shortly *diagram category*. For example, if $\mathcal{D} = \mathrm{Top}$ we write $\mathcal{C}$-Top for this category. We call $\mathcal{C}$ the *index category* of these diagrams. There are several examples we want to keep in mind.

Example 6.5.2. Let $\mathcal{G}$ be the category of a group G. Then a functor $F\colon \mathcal{G} \to \mathcal{D}$ is a G-object in $\mathcal{D}$. Thus G-objects in a category D are nothing else than $\mathcal{G}$-diagrams in $\mathcal{D}$. For instance, if $\mathcal{D}$ is the category of topological spaces we get a space with an action of G on the underlying set of the space by homeomorphisms. Recall, that if G is a topological group we do not necessarily get a G-space, because the action does not have to be continuous. Of course if G is equipped with the discrete topology we always get a G-space and in this case $\mathcal{G}$-diagrams of spaces and G-spaces are the same.

Example 6.5.3. Let G be a group and or(G) its orbit category. Let X be a G-space. The fixed point functor is an example of a contravariant or(G)-space. We called it the *associated contravariant* or(G)*-space* of X. The quotient space functor is an example of a covariant or(G)-space. A functor from or(G) to the category of spectra is an or(G)-spectrum. In Section 4.1 equivariant homology and cohomology theories are constructed from covariant (or respectively contravariant) or(G)-spectra.

Example 6.5.4. Let Δ be the category with objects the finite ordered sets $[n] := \{0, \ldots, n\}$ and morphisms the monotonically increasing maps. Then a *simplicial set* is a contravariant Δ-set. Let Δ_{inj} be the category with the same objects as Δ but with only the injective monotonically increasing maps. A *semisimplicial set* is a contravariant Δ_{inj}-set.

Definition 6.5.5. Let X be a contravariant and Y a covariant $\mathcal{C}$-space. Then their *balanced product* is defined to be the space

$$X \times_{\mathcal{C}} Y := \coprod_{C \in ob(\mathcal{C})} X(C) \times Y(C)/\sim$$

where $\sim$ is the equivalence relation generated by $(x\phi, y) \sim (x, \phi y)$ for morphisms $\phi\colon C \to D$ in $\mathcal{C}$ and $x \in X(D)$ and $y \in Y(C)$ where $x\phi$ stands for $X(\phi)(x)$ and ϕy for $Y(\phi)(y)$.

If X and Y are pointed, that is X is a contravariant and Y a covariant functor from $\mathcal{C}$ to the category of pointed spaces, then one defines their *balanced smash product* analogously to be the pointed space

$$X \wedge_{\mathcal{C}} Y = \bigvee_{C \in ob(\mathcal{C})} X(C) \wedge Y(C)/\sim .$$

If X is a contravariant pointed $\mathcal{C}$-space and E a covariant $\mathcal{C}$-spectrum, then their *balanced smash product* is defined level-wise to be $X \wedge_{\mathcal{C}} E_n$ together with the obvious maps.

Definition 6.5.6. Let X and Y be $\mathcal{C}$-spaces (of the same variance). Then we define $\mathrm{Map}_{\mathcal{C}}(X, Y)$ to be the space one gets by giving the set of maps of $\mathcal{C}$-spaces from X to Y the subspace topology of the inclusion into $\prod_{C \in ob(\mathcal{C})} \mathrm{Map}(X(C), Y(C))$.

If X is a pointed $\mathcal{C}$-space and E a $\mathcal{C}$-spectrum, then one defines the *mapping space spectrum* $\mathrm{Map}_{\mathcal{C}}(X, E)$ to be the spectrum whose space at level n is $\mathrm{Map}_{\mathcal{C}}(X, E(n))$ together with the structure maps coming from the canonical map of pointed spaces

$$\mathrm{Map}_{\mathcal{C}}(X, E(n)) \wedge S^1 \to \mathrm{Map}_{\mathcal{C}}(X, E(n) \wedge S^1),$$

which takes $\phi \wedge z$ to the map of $\mathcal{C}$-spaces from X to $E(n) \wedge S^1$ which sends $x \in X(C)$ to $\phi(C)(x) \wedge z \in E(n)(C) \wedge S^1$ for every object C of $\mathcal{C}$.

Remark 6.5.7. In [DL98] the balanced product is called *tensor product* and it is also denoted by $X \otimes_{\mathcal{C}} Y$. Note that the balanced product and the mapping space are also called the coend and the end construction in category theory (see [ML98, IX.5 and IX.6]).

There is an exponential law as expected:

Lemma 6.5.8 (Lemma 1.5 of [DL98])**.** *Let X be a contravariant $\mathcal{C}$-space, Y a covariant $\mathcal{C}$-space and Z a space. Write* $\mathrm{Map}(Y, Z)$ *for the obvious contravariant $\mathcal{C}$ -space whose value at an object C of $\mathcal{C}$ is* $\mathrm{Map}(Y(C), Z)$*. Then there is a homeomorphism*

$$\mathrm{Map}(X \times_{\mathcal{C}} Y, Z) \to \mathrm{Map}_{\mathcal{C}}(X, \mathrm{Map}(Y, Z))$$

which is natural in X, Y and Z. □

Next we introduce evaluation functors and the notation we will use for them.

Definition 6.5.9. Let $j\colon \mathcal{C}' \to \mathcal{C}$ be a functor. The *evaluation functor* at j is defined by the composition with j:

$$\mathrm{ev}_j\colon \mathcal{D}^{\mathcal{C}} \to \mathcal{D}^{\mathcal{C}'}, \quad F \mapsto F \circ j$$

An object C of $\mathcal{C}$ defines a subcategory with one object C and one morphism id_C. We write ev_C for the evaluation of the inclusion of this category and call it the *evaluation at C*. Explicitly it is given by

$$\begin{aligned} \mathrm{ev}_C\colon \mathcal{D}^{\mathcal{C}} &\to \mathcal{D} \\ F &\mapsto F(C) \\ (F \to G) &\mapsto (F(C) \to G(C)). \end{aligned}$$

If $g\colon C_1 \to C_2$ is a morphism in $\mathcal{C}$ and I the category with two objects 0 and 1, their identities, and one morphism $0 \to 1$, we have a functor $j : I \to \mathcal{C}$ which takes 0 to C_1, 1 to C_2 and $(0 \to 1)$ to g. Then we write ev_g for the evaluation of this functor and call it the *evaluation at the morphism g*.

Remark 6.5.10. If $\mathcal{D}'$ is another category and $F\colon \mathcal{D}' \to \mathcal{D}^{\mathcal{C}}$ a functor, then for every object C of $\mathcal{C}$ composition with the evaluation yields a functor $F_C = \mathrm{ev}_C \circ F$. Then for a morphism $g\colon C_1 \to C_2$ in $\mathcal{C}$ the evaluation ev_g provides a natural transformation

$$\mathrm{ev}_g \circ F : F_{C_1} \to F_{C_2}.$$

Therefore evaluation defines a functor

$$\begin{aligned} \mathrm{ev}\colon \mathcal{C} &\to \mathrm{Fun}(\mathcal{D}^{\mathcal{C}}, \mathcal{D}) \\ C &\mapsto \mathrm{ev}_C \\ g &\mapsto \mathrm{ev}_g, \end{aligned}$$

which we call the *evaluation functor*.

References

[BRS76] S. Buoncristiano, C. P. Rourke, and B. J. Sanderson, *A geometric approach to homology theory*, Cambridge University Press, Cambridge, 1976, London Mathematical Society Lecture Note Series, No. 18. MR MR0413113 (54 #1234)

[DF87] E. Dror Farjoun, *Homotopy and homology of diagrams of spaces*, Algebraic topology (Seattle, Wash., 1985), Lecture Notes in Math., vol. 1286, Springer, Berlin, 1987, pp. 93–134. MR MR922924 (89b:55012)

[DL98] J. F. Davis and W. Lück, *Spaces over a category and assembly maps in isomorphism conjectures in K- and L-theory*, K-Theory **15** (1998), no. 3, 201–252. MR MR1659969 (99m:55004)

[HSS00] M. Hovey, B. Shipley, and J. Smith, *Symmetric spectra*, J. Amer. Math. Soc. **13** (2000), no. 1, 149–208. MR MR1695653 (2000h:55016)

[Kas95] C. Kassel, *Quantum groups*, Graduate Texts in Mathematics, vol. 155, Springer-Verlag, New York, 1995. MR MR1321145 (96e:17041)

[KL05] M. Kreck and W. Lück, *The Novikov conjecture*, Oberwolfach Seminars, vol. 33, Birkhäuser Verlag, Basel, 2005, Geometry and algebra. MR MR2117411 (2005i:19003)

[Kre84] M. Kreck, *Bordism of diffeomorphisms and related topics*, Lecture Notes in Mathematics, vol. 1069, Springer-Verlag, Berlin, 1984, With an appendix by Neal W. Stoltzfus. MR MR755877 (86b:57015)

[LM09] G. Laures and J. E. McClure, *Multiplicative properties of Quinn spectra*, Arxiv.org 0907.2367 (http://arxiv.org/abs/0907.2367) (2009).

[Lüc05] W. Lück, *Equivariant cohomological Chern characters*, Internat. J. Algebra Comput. **15** (2005), no. 5-6, 1025–1052. MR MR2197819 (2007d:55008)

[May96] J. P. May, *Equivariant homotopy and cohomology theory*, CBMS Regional Conference Series in Mathematics, vol. 91, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1996, With contributions by M. Cole, G. Comezaña, S. Costenoble, A. D. Elmendorf, J. P. C. Greenlees, L. G. Lewis, Jr., R. J. Piacenza, G. Triantafillou, and S. Waner. MR MR1413302 (97k:55016)

[ML98] S. Mac Lane, *Categories for the working mathematician*, second ed., Graduate Texts in Mathematics, vol. 5, Springer-Verlag, New York, 1998. MR MR1712872 (2001j:18001)

[MMSS01] M. A. Mandell, J. P. May, S. Schwede, and B. Shipley, *Model categories of diagram spectra*, Proc. London Math. Soc. (3) **82** (2001), no. 2, 441–512. MR MR1806878 (2001k:55025)

[Qui95] F. Quinn, *Assembly maps in bordism-type theories*, Novikov conjectures, index theorems and rigidity, Vol. 1 (Oberwolfach, 1993), London Math. Soc. Lecture Note Ser., vol. 226, Cambridge Univ. Press, Cambridge, 1995, pp. 201–271. MR MR1388303 (97h:57055)

[RS72] C. P. Rourke and B. J. Sanderson, *Introduction to piecewise-linear topology*, Springer-Verlag, New York, 1972, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 69. MR MR0350744 (50 #3236)

[tD87] T. tom Dieck, *Transformation groups*, de Gruyter, 1987.

[tD00] ———, *Topologie*, 2 ed., de Gruyter, 2000.

[Wei94] C. A. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994. MR MR1269324 (95f:18001)

[Whi78] G. W. Whitehead, *Elements of homotopy theory*, Graduate Texts in Mathematics, vol. 61, Springer-Verlag, New York, 1978. MR MR516508 (80b:55001)

Erklärung

Hiermit versichere ich, dass ich diese Arbeit selbstständig verfasst, keine anderen als die genannten Quellen und Hilfsmittel benutzt, und Zitate kenntlich gemacht habe.