TOWARDS AN EXPANSION PRINCIPLE FOR TOPOLOGICAL AUTOMORPHIC FORMS

SEBASTIAN THYSSEN

Abstract. The theory of topological automorphic forms is intimately connected to the theory of unitary group Shimura varieties of type $U(n-1,1)$. In this note we consider certain integral models of such Shimura varieties. We introduce an expansion of automorphic forms by considering their pullbacks to the formal completion along a special divisor. Unfortunately, the use of divided powers debilitates the expansion principle for this expansion.

Contents

1. Introduction

The theory of topological automorphic forms can be considered as a generalization of a p-complete version of topological modular forms to higher dimensions. One classifies higher dimensional abelian schemes which are equipped with extra structure to make the moduli problem usable in topology.

This extra structure takes care of the fact that, so far, topology is tied exclusively to the study of one dimensional formal groups, in that it breaks up the p-divisible group of the classified abelian schemes in a way that their deformation theory is controlled by a one-dimensional summand. It is this fact which makes Lurie's realization theorem applicable to the underlying moduli stack.

In the case of elliptic curves, i.e. one-dimensional abelian schemes, no such extra structure is needed as the dimension of the group schemes equals the dimension of the associated formal groups. One can employ Lurie's theorem to construct p-complete versions of topological modular forms, but those can be assembled via an arithmetic square into an integral or arithmetically global spectrum. In other words the construction of TMF does not rely on or better does not stop with Lurie's theorem. The moduli stack of elliptic curves is defined over $\operatorname{Spec} \mathbb{Z}$.

The restriction to a single prime as in the case of topological automorphic forms comes from imposing the extra structure on the abelian schemes. More precisely, the extra data requires a choice of number field k and the construction of a TAF-type spectrum is henceforth restricted to primes p in k which are split. Since there is no number field in which all primes are split the idea to use an arithmetic square to assemble TAF-spectra for different primes breaks down and the construction ends with the realization theorem.

The purpose of this report on an ongoing project is to promote and continue the study of the theory of topological automorphic forms. We consider certain integral models of Shimura varieties of type $U(n-1,1)$ and a special divisor on them. Note that the definition of these structures makes sense over $\operatorname{Spec}\mathcal{O}_k$, the spectrum of the ring of integers of any chosen number field k. The actual topological realization then makes sense once we pull these structures back to $\operatorname{Spec}\mathbb{Z}_p$. Note that the p-adic numbers arise from the completion map $\mathcal{O}_k \to \mathcal{O}_{k,u^c} \cong \mathbb{Z}_p$ where the primes of interest splits as $p = uu^c$.

The divisor $\mathcal{Z}$ is defined as the moduli stack where the n-dimensional abelian schemes A_n as classified by the integral model $\mathcal{M}$ of the Shimura variety factor as $A_n \cong A_{n-1} \times E$ into the product of an $(n-1)$-dimensional scheme and an elliptic curve. We consider the formal completion $\widehat{\mathcal{M}}_{\mathcal{Z}}$ of the integral model along the divisor and show that it satisfies the conditions of Lurie's theorem.

Theorem. *$\widehat{\mathcal{M}}_{\mathcal{Z}}$ is derived in the sense that the realization problem*

$$\widehat{\mathcal{M}}_{\mathcal{Z}} \to \mathcal{M}^n_{p-div} \to \mathcal{M}_{fg}$$

has a canonical solution.

An immediate consequence of this result is the existence of an E_∞-ring spectrum X, as the global sections of $\mathcal{O}^{top}_{\widehat{\mathcal{M}}_{\mathcal{Z}}}$. By functoriality of Lurie's theorem X carries a map

$$TAF_n \to X$$

induced by the completion map $\widehat{\mathcal{M}}_{\mathcal{Z}} \to \mathcal{M}$.

We immediately remark that this procedure is of course iterable. Hence we obtain a sequence of maps

$$TAF_n \to X := X_1 \to X_2 \to \cdots \to X_{n-1},$$

where X_{n-1} is the E_∞-ring associated to the moduli stack classifying the n-fold product of elliptic curves considered as divisor in $\mathcal{M}$ of codimension $n-1$. Note that the since dimension of $\mathcal{M}$ is $n-1$ the points of X_{n-1} have complex multiplication.

As in the case of TAF_n, Lurie's theorem provides the existence of a descent spectral sequence (conditionally) converging to π_*X and of course, we would like to determine the homotopy of X and understand the homotopy of the map $TAF_n \to X$, unfortunately this is out of reach. Thus, in the rest of the document we try to understand this map after applying edge homomorphisms of the descent spectral sequences on both sides, e.g. $\pi_*TAF_n \xrightarrow{edge} H^0(\mathcal{M}, \omega^{\otimes *})$. That is, we study

$$H^0(\mathcal{M}, \omega^{\otimes *}) \to H^0(\widehat{\mathcal{M}}_{\mathcal{Z}}, \omega^{\otimes *})$$

and how to interpret the right hand side object.

It turns out that the global sections of the invertible sheaf $\omega^{\otimes l}$ on the formal completion

$$H^0(\widehat{\mathcal{M}}_{\mathcal{Z}}, \omega^{\otimes l}) \subseteq H^0(\widehat{\mathcal{M}}^{pd}_{\mathcal{Z}}, \omega^{\otimes l})$$

include into the global sections of a rather different kind of completion, the divided power completion $\widehat{\mathcal{M}}^{pd}_{\mathcal{Z}}$ of $\mathcal{M}$ along the divisor $\mathcal{Z}$. We are able to show that global sections of $\omega^{\otimes l}$ on the divided power completion behave like power series of sections (of shifted powers of ω) on the divisor.

Theorem. *At odd primes p we have,*

$$H^0(\widehat{\mathcal{M}}^{pd}_{\mathcal{Z}}, \omega^{\otimes l}) \cong H^0(\mathcal{Z}, \omega^{\otimes l}\langle\langle \omega \otimes \omega_E \rangle\rangle).$$

Put in different words, a section of $\omega^{\otimes l}$ on the divided power completion, and thus on the ordinary formal completion, can be interpreted as family of sections of (shifted powers of) ω on the divisor. The sheaf ω_E is denoted to be precise, but can be ignored as it is a constant sheaf. For more details we refer the reader to Section 4.2.

There are two major inputs to this result. The first is the existence of a retraction from the divided power completion to the divisor providing a section $\mathcal{O}_Z \to \mathcal{O}_{\widehat{\mathcal{M}}^{pd}_{\mathcal{Z}}}$ to the usual projection.

Theorem. *Z is a retract of $\widehat{\mathcal{M}}^{pd}_Z$. That is, there is a canonical retraction r*

$$\begin{tikzcd} Z \arrow[r] & \widehat{\mathcal{M}}^{pd}_Z . \arrow[l, bend right, dashed, "r"'] \end{tikzcd}$$

The second is the existence of a section of the normal bundle of the divisor over the divided power completion. Both results use Grothendieck-Messing deformation theory.

We show that the map $Z \to \mathcal{M}$ is surjective on π_0. As corollary we derive that sections of ω on the ordinary formal completion inject into families of sections of shifted powers of ω on the divisor.

Corollary. *At odd primes there is an injective ring homomorphism*

$$T\colon\ \mathrm{H}^0\big(\mathcal{M}, \omega^{\otimes k}\big) \longrightarrow \mathrm{H}^0\big(Z, \omega^{\otimes k}\langle\langle \omega \otimes \omega_E \rangle\rangle\big)\,.$$

We interpret the map T as expansion of automorphic forms. For $\mathbb{Z}_p$-algebras of characteristic zero we are able to show T remains injective after tensoring up. Unfortunately, this is no longer true for $\mathbb{Z}_p$-modules with torsion. A consequence of this is a very limited expansion principle.

Proposition. *Let $L_1 \subseteq L_2$ be $\mathbb{Z}_p$-algebras of characteristic 0 with torsion-free quotient. Then the diagram*

$$\begin{tikzcd} \mathrm{H}^0\big(\mathcal{M}_{/L_1}, \omega^{\otimes k}\big) \arrow[r, "T_{L_1}"] \arrow[d] & \mathrm{H}^0\big(Z_{/L_1}, \omega^{\otimes k}\langle\langle \omega \otimes \omega_E \rangle\rangle\big) \arrow[d] \\ \mathrm{H}^0\big(\mathcal{M}_{/L_2}, \omega^{\otimes k}\big) \arrow[r, "T_{L_2}"] & \mathrm{H}^0\big(Z_{/L_2}, \omega^{\otimes k}\langle\langle \omega \otimes \omega_E \rangle\rangle\big) \end{tikzcd}$$

is a pullback.

The torsion condition is necessary, since the inclusion $\mathcal{O}_{\widehat{\mathcal{M}}_Z} \to \mathcal{O}_{\widehat{\mathcal{M}}^{pd}_Z}$ of sheaves of rings is unfortunately not well behaved mod p. We see this locally, where the ideal sheaves are generated by a single generator x. Then $x^p = p! \cdot x^{[p]}$ maps to zero and this map, hence also T, is no longer an inclusion.

This defect suggest that it would be preferable to use the ordinary formal completion instead of the divided power completion. Unfortunately, we are not in the position to prove a statement like the interpretations of sections as families of sections on the divisor in this setting. We refer the reader to Section 4.3 for more details.

Acknowledgements. We would like to thank Ulrich Görtz, David Hansen, Benjamin Howard, Tyler Lawson, Martin Olbermann and Peter Scholze for helpful conversations and the DFG SPP 1786 for financial support.

2. Integral models of Shimura varieties and a divisor

Let k be a quadratic imaginary number field and $\mathcal{O}_k$ its ring of integers. We are interested in primes p which are split $p = uu^c$ in $\mathcal{O}_k$, so that $\mathcal{O}_{k,p} \cong \mathbb{Z}_p \oplus \mathbb{Z}_p$, where $-^c$ denotes the restriction of complex conjugation for a fixed embedding of k into the complex numbers.

2.1. **Integral models of unitary Shimura varieties.** For all integers $n, r \in \mathbb{N}_0$ such that $r \leq n$, we denote by $\mathcal{M}_{(n-r,r)}$ the stack associating to a locally noetherian $\mathcal{O}_k$-scheme S the groupoid $\mathcal{M}_{(n-r,r)}(S)$ of triples (A, i, λ) consisting of

(1) an n-dimensional abelian scheme A over S together with
(2) an action
$$i : \mathcal{O}_k \longrightarrow \mathrm{End}(A)$$
of $\mathcal{O}_k$ on A satisfying a $(n-r,r)$-*signature condition*: Locally on S, the characteristic polynomial of the induced action on Lie_A is
$$\mathrm{char}(T, i_*(a)|_{\mathrm{Lie}_A}) = (T - \varphi_S(a))^{n-r}(T - \varphi_S(\bar{a}))^r \in \mathcal{O}_S[T]\,,$$
for all $a \in \mathcal{O}_k$ and $\varphi_S : \mathcal{O}_k \to \mathcal{O}_S$ the structure morphism. The action i is required to be compatible with
(3) an $\mathcal{O}_k$-linear principal polarization
$$\lambda : A \longrightarrow A^\vee.$$

The compatibility condition imposed on action and polarization is that the Rosati involution
$$\psi \mapsto \psi^* = \lambda^{-1} \circ \psi^\vee \circ \lambda$$
on $\mathrm{End}\, A \otimes \mathbb{Q}$ induces the non-trivial automorphism
$$i(a)^* = i(a^c)$$
of $\mathcal{O}_k$. To further motivate the signature condition note that loosely speaking, this condition means that the representation of the induced action is equivalent to $n-r$ copies of the natural representation and r copies of the conjugate representation. Hence, locally Lie_A behaves like the $\mathcal{O}_k \otimes_{\mathbb{Z}} \mathcal{O}_S$-module $\mathcal{O}_S^{n-r} \oplus \mathcal{O}_S^r$, where $\mathcal{O}_k$ acts on the factor $\mathcal{O}_S^{n-r}$ through the structure morphism
$$\mathcal{O}_k \xrightarrow{\varphi_S} \mathcal{O}_S$$
and on the factor $\mathcal{O}_S^r$ through the conjugate of the structure morphism.

A morphism in $\mathcal{M}_{(n-r,r)}(S)$ from (A, λ, i) to (A', λ', i') is an $\mathcal{O}_k$-linear isomorphism
$$\alpha : A \longrightarrow A'$$
such that
$$\alpha^*(\lambda') = \lambda.$$

Remark. As pointed out by Pappas the stacks $\mathcal{M}_{(n-r,r)}$ are in general neither flat nor regular over $\mathrm{Spec}\, \mathcal{O}_k$ at ramified primes. One can amend this by imposing further conditions [11], [7], but this will not be of any interest to us as we are only interested in unramified places.

In fact, our situation is covered by the following

Proposition 2.1 ([8], Proposition 2.1)**.** *$\mathcal{M}_{(n-r,r)}$ is a locally Noetherian, separated Deligne-Mumford stack over $\operatorname{Spec}\mathcal{O}_k$. Furthermore, if we invert the discriminant Δ_k of k in $\mathcal{O}_k$, then*

$$\mathcal{M}_{(n-r,r)} \times_{\mathcal{O}_k} \operatorname{Spec}\mathcal{O}_k[\Delta_k^{-1}]$$

is smooth of relative dimension $r \cdot (n-r)$ over $\operatorname{Spec}\mathcal{O}_k[\Delta_k^{-1}]$. □

We provide a simple example of such moduli stacks.

Example 2.2. The moduli stack $\mathcal{M}_{(1,0)}$ of elliptic curves E with complex multiplication by $\mathcal{O}_k$ is finite and étale over $\operatorname{Spec}\mathcal{O}_k$. In particular, it is étale over the coarse moduli stack which is given by the scheme $\operatorname{Spec}\mathcal{O}_H$ where H is the Hilbert class field of K. Note that in this case $\operatorname{End}_{\mathcal{O}_k}(E, i_0) \cong \mathcal{O}_k$, which means i_0 is an isomorphism, and further, that the induced action

$$\mathcal{O}_k \longrightarrow \operatorname{End}_{\mathcal{O}_S}(\operatorname{Lie}_E) \cong \mathcal{O}_S$$

on the Lie algebra of the elliptic curve coincides with the structure homomorphism. An instance of this can be found in [9].

In order to keep notation simple we introduce the following notation

Notation 2.3. Let us denote the universal object over $\mathcal{M}_{(n-r,r)}$ by A_n or in the case of elliptic curves by E.

2.2. **A special arithmetic divisor.** In this section we describe a special arithmetic divisor, not of $\mathcal{M}_{(n-r,r)}$ itself, but of the pullback

$$\mathcal{M}_{(n-r,r)} \times_{\mathcal{O}_k} \mathcal{M}_{(1,0)}.$$

When S is connected there is a free $\mathcal{O}_k$-module $\operatorname{Hom}_{\mathcal{O}_k}(E, A_n)$ of finite rank for every pair (A_n, E) in the groupoid $(\mathcal{M}_{(n-r,r)} \times_{\mathcal{O}_k} \mathcal{M}_{(1,0)})(S)$. On this $\mathcal{O}_k$-module there is an $\mathcal{O}_k$-valued hermitian form given by

$$\langle f, g\rangle := i_E^{-1}(\lambda_E^{-1} \circ g^\vee \circ \lambda_{A_n} \circ f) \in \mathcal{O}_k \cong \operatorname{End}_{\mathcal{O}_k}(E, i_E).$$

Definition 2.4 ([6], Definition 3.2.1.)**.** For any $m \neq 0$ the Kudla-Rapoport divisor $\operatorname{KR}(m)$ is the moduli stack of triples (A_n, E, f) over $\mathcal{O}_k$-schemes S, such that

(1) $(A_n, E) \in (\mathcal{M}_{(n-r,r)} \times_{\mathcal{O}_k} \mathcal{M}_{(1,0)})(S)$, and
(2) $f \in \operatorname{Hom}_{\mathcal{O}_k}(E, A_n)$ satisfies $\langle f, f\rangle = m$.

As mentioned in [8], the cycles $\operatorname{KR}(m)$ are defined for arbitrary partitions of n, but are particularly interesting for $r = 1$. In this case they are divisors of codimension m in $\mathcal{M}_{(n-1,1)} \times_{\mathcal{O}_k} \mathcal{M}_{(1,0)}$.

Lemma 2.5. *For $r = m = 1$ the stacks $\operatorname{KR}(m)$ and $\mathcal{M}_{(n-r-1,r)} \times_{\mathcal{O}_k} \mathcal{M}_{(1,0)}$ are isomorphic as stacks over $\mathcal{M}_{(n-r,r)} \times_{\mathcal{O}_k} \mathcal{M}_{(1,0)}$.*

Proof. In any case the stacks $\operatorname{KR}(m)$ and $\mathcal{M}_{(n-r-1,r)} \times_{\mathcal{O}_k} \mathcal{M}_{(1,0)}$ are stacks over $\mathcal{M}_{(n-r,r)} \times_{\mathcal{O}_k} \mathcal{M}_{(1,0)}$. The former by the forgetful map, the latter by taking the product. Let us consider the case $r = m = 1$, then

$$\begin{tikzcd}
\operatorname{KR}(1) \arrow[r, "\cong"] \arrow[dr] & \mathcal{M}_{(n-2,1)} \times_{\mathcal{O}_k} \mathcal{M}_{(1,0)} \arrow[d] \\
& \mathcal{M}_{(n-1,1)} \times_{\mathcal{O}_k} \mathcal{M}_{(1,0)}
\end{tikzcd}$$

as follows. Starting with a triple $(A_n, E, f) \in \mathrm{KR}(1)(S)$ the condition $m = 1$ means that f is the inclusion of a direct summand or likewise that the kernel

$$\ker(f^\vee \colon A_n \longrightarrow E)$$

of the dual isogeny is connected. Since the map f is required to be compatible with all extra structures of E and A_n, both arguments provide a splitting

$$A_n \cong A_{n-1} \times E$$

and thus a point (A_{n-2}, E) of $\mathcal{M}_{(n-2,1)} \times_{\mathcal{O}_k} \mathcal{M}_{(1,0)}$. This is the inverse to the obvious map in the other direction, where (A_{n-1}, E) gets mapped to $(A_{n-1} \times E, E, i \colon E \to A_{n-1} \times E)$. □

In order to keep notation simple we introduce the following notations.

Notation 2.6. We remind the reader that $\mathcal{O}_{k,u^c} \cong \mathbb{Z}_p$.

(1) For the rest of the document the notation $\mathcal{M}_{(n-r,r)}$ will denote the moduli stacks
$$\mathcal{M}_{(n-r,r)} \times_{\mathcal{O}_k} \operatorname{Spec} \mathcal{O}_{k,u}$$
over $\operatorname{Spec} \mathcal{O}_{k,u}$ unless mentioned otherwise.

(2) Let further
$$\mathcal{M} := \mathcal{M}_{(n-1,1)} \times_{\mathcal{O}_k} \mathcal{M}_{(1,0)}.$$
In view of (1) this will be a stack over $\operatorname{Spec} \mathcal{O}_{k,u}$ unless otherwise mentioned.

(3) For $r = 1$ we will denote the Kudla-Rapoport divisor KR(1) by
$$Z := \mathrm{KR}(1)$$
since it will frequently appear as index. Unless otherwise mentioned this notation shall again denote the pullback to $\operatorname{Spec} \mathcal{O}_{k,u}$.

(4) We denote by A the universal object (A_n, E) over $\mathcal{M}$ and by A_0 the universal object $A_{n-1} \times E$ over Z.

2.3. **Splittings, polarizations and interactions.** In this section we explain why the theory of topological automorphic forms is a p-complete theory. We shed some light on the necessity of structures the abelian schemes have to be equipped with. In particular, the existence of the action and the polarization and the interaction of both have important consequences for our main objects of study.

As before let A_n be the universal object over $\mathcal{M}_{(n-1,1)}$ and A the universal object over $\mathcal{M}$. The endomorphisms of the p-divisible group $A_n(p)$ of A_n are naturally p-complete, since they are the limit

$$\operatorname{End} A_n(p) = \lim_k \operatorname{Hom}(A_n[p^k], A_n(p))$$

over p^k-torsion modules $\operatorname{Hom}(A_n[p^k], A_n(p))$. Hence the induced action

$$i_* \colon \mathcal{O}_k \longrightarrow \operatorname{End} A_n(p)$$

factors over $\mathcal{O}_{k,p} \cong \mathcal{O}_{k,u} \oplus \mathcal{O}_{k,u^c}$ and thus gives a splitting of the p-divisible group

$$A_n(p) \cong A_n(u) \oplus A_n(u^c). \tag{1}$$

To split the p-divisible group is a good first step. However, this splitting gives a priori no control over the dimensions of the summands.

Lemma 2.7. *Let S be a scheme such that p is locally nilpotent on S. Then the natural map of $\mathcal{O}_K$-linear objects*

$$A(p) \longrightarrow A$$

induces a natural isomorphism on the level of Lie algebras

$$\mathrm{Lie}_{A(p)} \xrightarrow{\cong} \mathrm{Lie}_A.$$

Proof. Since A is smooth over S with identity section we can choose an affine chart $\operatorname{Spec} R$ on S so that we can pick coordinates $x_1, \cdots, x_n$ in a neighborhood U of the identity in A, such that these coordinates vanish on the identity. The identity section is given by the ideal $\mathfrak{m} = (x_1, \cdots, x_n)$, hence if we complete $U = \operatorname{Spec} R[x_1, \cdots, x_n]$ at the identity section in this chart we get

$$\widehat{U} = \operatorname{Spf} R[\![x_1, \cdots, x_n]\!] .$$

Since A is an n dimensional group we also get an n dimensional formal group law on $R[\![x_1, \cdots, x_n]\!]$. The p-series given by

$$[p](x_1, \ldots, x_n) = (f_1(x_1, \ldots, x_n), \ldots, f_n(x_1, \ldots, x_n)).$$

Note that the scheme theoretic kernel of the multiplication-by-p-map represented by the p-series is given by the ideal $(f_1, \ldots, f_n) \in R[x_1, \ldots, x_n]$. By definition of group laws

$$f_i(x_1, \cdots, x_n) \equiv px_i \bmod \deg 2.$$

Hence by the same arguments as in the one dimensional case one finds that mod p the p-series consists only of terms of degree p^h and higher. Analogously, for $[p^2]$ we find that p^2 divides all coefficients up to degree p^h and further that p divides all coefficients up to order p^{2h}. Consequently, $[p^2]$ mod p consists only of terms of degree p^{2h} and higher.

This generalizes nicely, so that if $p^k = 0$ in R, then $[p^{ik}]$ consists only of terms of degree p^{ikh} and higher. In particular, if we denote by I^k the ideal generated by $[p^k]$, then since I^k is contained in a power of the maximal ideal $\mathfrak{m}$ given by the coordinates, more precisely

$$I^k \subseteq \mathfrak{m}^{p^{kh}},$$

we see that $\mathrm{Inf}_U^{p^{kh}}(S) \subseteq A[p^k]$ and thus $\widehat{A} \subset A(p)$ in the colimit.

It is clear that in this chart

$$A[p^k] = \operatorname{Spec} R[[x_1, \cdots, x_n]]/I^k.$$

Furthermore, the maximal ideal of the local ring $R[[(x_1, \cdots, x_n)]]/I^k$ is given by $\mathfrak{m}/I^k$ and thus the cotangent space is $(\mathfrak{m}/I^k)/(\mathfrak{m}/I^k)^2$. Note that the Lie algebra of $A[p^k]$ in this chart is the dual of this quotient. Since $[p^k] \subseteq \mathfrak{m}^2$ the isomorphism theorem from linear algebra tells us that

$$(\mathfrak{m}/I^k)/(\mathfrak{m}/I^k)^2 \cong \mathfrak{m}/\mathfrak{m}^2.$$

By dualizing we find hence that locally

$$\mathrm{Lie}_{A[p^k]} \xrightarrow{\cong} \mathrm{Lie}_A$$

and thus

$$\mathrm{Lie}_{A(p)} \xrightarrow{\cong} \mathrm{Lie}_A$$

by taking the colimit again.

□

By the lemma and the definition of dimension of a p-divisible group it is now clear that the $(n-1,1)$-signature condition imposed on the abelian schemes determines the dimension of the summands in the splitting of $A_n(p)$:

$$A_n(p) \cong \underbrace{A_n(u)}_{(n-1)-dim} \oplus \underbrace{A_n(u^c)}_{1-dim}.$$

The action i of $\mathcal{O}_K$ is required to take the complex conjugation in $\mathcal{O}_K$ to the Rosati involution. Therefore the isomorphism

$$\lambda_*\colon A_n(p) \xrightarrow{\cong} A_n^\vee(p)$$

induced by the polarization λ of A breaks up into

$$A_n(u) \xrightarrow{\cong} A_n^\vee(u^c)$$
$$A_n(u^c) \xrightarrow{\cong} A_n^\vee(u).$$

One defines ω_{A_n} as pullback of the of sheaf of relative differentials of $A_n(p)$ over $\mathcal{M}_{(n-1,1)}$ to the latter. There is an obvious (induced) splitting

$$\omega_{A_n} \cong \omega_{A_n}^+ \oplus \omega_{A_n}^-.$$

Note that by the lemma ω_{A_n} is also the dual of Lie_{A_n}. Hence

$$\mathrm{Lie}_{A_n(u^c)} \cong \mathrm{Lie}_{A_n} \otimes_{\mathcal{O}_{k,p}} \mathcal{O}_{k,u^c} =: \mathrm{Lie}_{A_n}^-,$$

determines $\omega_{A_n}^-$ to be locally free of rank one. If the context is clear we may drop the dimension n of the abelian schemes from notation and just write ω_A, ω_A^+ and ω_A^-, respectively.

2.4. **Grothendieck-Messing theory.** By Grothendieck-Messing deformation theory there is a contravariant Dieudonné crystal $\mathbb{D}(X_0)$ for any p-divisible group X_0 over a commutative ring R_0. To each locally nilpotent divided power (or for short *pd*-extension, coming from the french "puissances divisées") extension $(R, I, (\gamma_i))$, which henceforth will be denoted just by R, there is a functorially attached locally free R-module $\mathbb{D}(X_0)_R$ of rank $\mathrm{ht}X_0$. If there is a p-divisible group X over R extending X_0, then there is a functorial short exact sequence

$$\mathrm{Lie}^\vee_{X_{/R}} \longrightarrow \mathbb{D}(X_0)_R \longrightarrow \mathrm{Lie}_{X^\vee_{/R}}.$$

If A is an abelian scheme over R such that there is an isomorphism

$$A(p) \times_R \mathrm{Spec}\, R_0 \cong X_0$$

of p-divisible groups over R_0, then there exists a natural isomorphism

$$\mathbb{D}(X_0)_R \cong \mathrm{H}^1_{\mathrm{dR}}(A_{/R})$$

between the module and the first de Rham cohomology of A over R. Further this isomorphism identifies the short exact sequence

$$\mathrm{Lie}^\vee_{A(p)_{/R}} \longrightarrow \mathbb{D}(A(p))_R \longrightarrow \mathrm{Lie}_{A(p)^\vee_{/R}}$$

with the Hodge filtration

$$\mathrm{Lie}^\vee_{A_{/R}} \longrightarrow \mathrm{H}^1_{\mathrm{dR}}(A_{/R}) \longrightarrow \mathrm{Lie}_{A^\vee_{/R}}.$$

Remark 2.8. It is worth mentioning that there is also a dual, i.e. covariant, theory of crystals. In case of p-divisible groups arising from abelian varieties one then compares the Dieudonné crystal to de Rham homology.

Remark 2.9. Since we will most frequently deal only with p-divisible groups coming from abelian schemes A we will simplify the notation and just write

$$\omega_{A/R} \longrightarrow \mathbb{D}(A)_R \longrightarrow \mathrm{Lie}_{A^\vee_{/R}}.$$

In view of the following sections let us remark, that we will frequently consider Dieudonné crystals of split p-divisible groups e.g. $A(p) \cong A(u) \oplus A(u^c)$. This induces a splitting

$$\mathbb{D}(A(p)) \cong \mathbb{D}(A(u)) \oplus \mathbb{D}(A(u^c))$$

with short exact sequences

$$\begin{aligned} \mathrm{Lie}^\vee_{A(u)_{/R}} &\longrightarrow \mathbb{D}(A(u))_R \longrightarrow \mathrm{Lie}_{A(u)^\vee_{/R}} \\ \mathrm{Lie}^\vee_{A(u^c)_{/R}} &\longrightarrow \mathbb{D}(A(u^c))_R \longrightarrow \mathrm{Lie}_{A(u^c)^\vee_{/R}} \end{aligned}$$

which we may sometimes refer to as

$$\mathbb{D}(A) \cong \mathbb{D}(A)^+ \oplus \mathbb{D}(A)^-$$

with short exact sequences

$$\omega^\pm_{A/R} \longrightarrow \mathbb{D}(A)^\pm_R \longrightarrow \mathrm{Lie}^\mp_{A/R}.$$

3. Deriving the formal completion

In this section we show that the formal completion $\widehat{\mathcal{M}}_Z$ is a derived formal stack.

Proposition 3.1. *The formal stack $\widehat{\mathcal{M}}_Z$ is derived, in the sense that the realization problem*

$$\widehat{\mathcal{M}}_Z \to \mathcal{M}^n_{p-div} \to \mathcal{M}_{fg}$$

has a canonical solution.

Proof. By Theorem 4.7 of [5] it is sufficient to show that there is a formally étale map $\widehat{\mathcal{M}}_Z \to \mathcal{M}^n_{p-div}$. Using work of Behrens and Lawson it suffices to show that there is a formally étale map

$$\widehat{\mathcal{M}}_Z \to \mathcal{M}.$$

The completion map is the obvious candidate. Completion maps are flat and, though they are not unramified as they are not locally of finite presentation, they are formally unramified, and hence formally étale. (Cf. also Corollary 4.4 of [1].) We win, since compositions of formally étale maps are formally étale.

□

Remark 3.2. In fact, the version of Lurie's theorem in [3] Behrens and Lawson use employs the Serre-Tate deformation property as condition and does not mention formally étale maps at all.

In Example 4.8 of [5] Goerss states that in an addendum to his realization result Lurie shows that if the map $\mathcal{M} \to \mathcal{M}^n_{p-div}$ from a locally noetherian DM-stack $\mathcal{M}$ to the moduli stack $\mathcal{M}^n_{p-div}$ of p-divisible groups of height exactly n satisfies Serre-Tate theorem, then it is formally étale.

Here is a sketch of why the map

$$\mathcal{M} \to \mathcal{M}^n_{p-div}$$

given by

$$(A, i, \lambda) \mapsto A(u^c)$$

satisfies the Serre-Tate deformation condition:

- the ordinary Serre-Tate theorem provides an equivalence of categories

$$\{ \text{ Def's of } A \ \} \simeq \{ \text{ Def's of } A(p) \ \}$$

- employing the action i we find

$$\{ \text{ Def's of } (A, i) \ \} \simeq \{ \text{ Def's of } A(p) \cong A(u) \times A(u^c) \ \}$$

  where the deformations on the right hand side respect the splitting. Hence

$$\{ \text{ Def's of } (A, i) \ \} \simeq \{ \text{ Def's of } A(u) \ \} \times \{ \text{ Def's of } A(u^c) \ \}$$

- using the isomorphism $\lambda_*\colon A(u^c) \cong A(u)^\vee$ induced by the polarization λ we see that a deformation of say $A(u^c)$ determines a deformation of $A(u)$. Hence

$$\{ \text{ Def's of } (A, i, \lambda) \ \} \simeq \{ \text{ Def's of } A(u^c) \ \}$$

As upshot we conclude that the extra structure the abelian schemes are equipped with allows the deformation theory of $A(p)$ to be completely controlled by the 1-dimensional summand $A(u^c)$.

By functoriality of Luries theorem as in [3] the completion map also induces a map on the level of E_∞-ring spectra

$$TAF_n \to X,$$

where TAF_n denotes the height n theory of topological modular forms associated to $\mathcal{M}$ as global sections of $\mathcal{O}^{top}_{\mathcal{M}}$ and X denotes the global sections of $\mathcal{O}^{top}_{\widehat{\mathcal{M}}_Z}$.

In the following sections we investigate the homotopy of this map.

4. A retraction and a section

In this section we collect existence results for retractions both from the divided power completion of $\mathcal{M}$ at the divisor $\mathcal{Z}$ and from the ordinary formal completion. We introduce a canonical section of a certain invertible sheaf, explain how the existence of such section is enables us to identify the structure sheaf of the formal completion as something like a sheaf of power series rings. Unfortunately, this section seems to exist only in the divided power case, which prevents us from using the retraction from the ordinary formal completion and forces us to use the retraction from the divided power completion in later sections.

4.1. **A retraction from the divided power completion.** In this section we prove that there is a canonical retraction r

$$\begin{tikzcd} Z \arrow[r] & \widehat{\mathcal{M}}_Z^{pd} \arrow[l, bend right, dashed, "r"'] \end{tikzcd}$$

from the divided power completion $\widehat{\mathcal{M}}_Z^{pd}$ of the moduli stack $\mathcal{M}$ along the Kudla-Rapoport divisor Z to the divisor.

First, we collect some commutative algebra. Let R be a ring and

$$I : I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$$

a fixed chain of ideals in R. We say R is complete with respect to the I-adic topology if there is an isomorphism

$$R \xrightarrow{\cong} \lim R/I_n.$$

This map is injective if and only if $\bigcap_{n=0}^{\infty} I_n = (0)$ and surjective if and only if R has all Cauchy sequences. If R is complete with respect to I, we call R an I-adic ring.

For an I-adic ring R any element a that maps to a unit under projection mod I_1 is already a unit in R, since for all $b \in I_1$ one finds that $1 + b$ is a unit in R.

Lemma 4.1. *Let R be an I-adic ring. If $f \colon R \longrightarrow R^{n+1}$ is given by*

$$1 \mapsto (a_1, \cdots, a_{n+1})$$

is a split injection and $a_{n+1} \in I_1$, then the composition

$$R \xrightarrow{f} R^{n+1} \xrightarrow{pr_n} R^n$$

given by $1 \mapsto (a_1, \cdots, a_n)$ of f with the projection onto the first n summands is already a split injection.

Proof. Being a split injection is equivalent to the ideal $(a_1, \cdots, a_{n+1})$ generating R, which means that there is a unit $u \in R$ and a linear combination

$$u = \lambda_1 a_1 + \cdots + \lambda_n a_n + \lambda_{n+1} a_{n+1}.$$

By assumption $\lambda_1 a_1 + \cdots + \lambda_n a_n + \lambda_{n+1} a_{n+1}$ maps to

$$\overline{u} = \overline{\lambda_1 a_1} + \cdots + \overline{\lambda_n a_n} + \overline{0}$$

under the projection mod I_1. All preimages of units in R/I_1 are units in R, hence $\lambda_1 a_1 + \cdots + \lambda_n a_n$ is already a unit in R and the ideal $(a_1, \cdots a_n)$ generates R. It follows that the map $1 \mapsto (a_1, \cdots a_n)$ is already a split injection.

□

Let S be a formal scheme. An affine T-point of S is an affine scheme $T = \operatorname{Spec} R$, such that for all opens U the map

$$\mathcal{O}_S(U) \longrightarrow R(U)$$

is a continuous map of adic rings. If S is a *pd*-formal scheme, then the map is required to be a continuous map of adic *pd*-rings. In particular, the ideal $I \subseteq R$ of definition has to have a *pd*-structure.

Let T be an affine point of $\widehat{\mathcal{M}_Z^{pd}}$ and T_0 its pullback to Z. Then the rank 2 crystal of the elliptic curve E has a special structure when evaluated on T_0. From the associated short exact sequences and signature condition one finds

$$\mathbb{D}(E)^-_{T_0} \cong \mathrm{Lie}^+_{E/T_0} \cong \mathrm{Lie}_{E/T_0}$$

and

$$\mathbb{D}(E)^+_{T_0} \cong \omega^+_{E/T_0} \cong \omega_{E/T_0}.$$

We are now in the position to show the main result of this section.

Theorem 4.2. *Z is a retract of $\widehat{\mathcal{M}}_Z^{pd}$. That is, there is a canonical retraction r*

$$\begin{tikzcd} Z \arrow[r] & \widehat{\mathcal{M}}_Z^{pd} \arrow[l, bend right, dashed, "r"'] \end{tikzcd}.$$

Proof. As before let A and A_0 denote the universal objects over $\widehat{M}_Z^{pd}$ and Z, respectively.

Consider an affine T-point of $\widehat{M}_Z^{pd}$ and its pullback T_0 to the divisor:

$$\begin{tikzcd} T_0 \arrow[r] \arrow[d] & Z \arrow[d] \\ T \arrow[r] & \widehat{M}_Z^{pd}. \end{tikzcd}$$

We will show that the bottom horizontal map lifts to Z. By functoriality all maps from affines T to $\widehat{M}_Z^{pd}$ will lift to Z compatibly, thus giving rise to the desired retraction map.

The crystal $\mathbb{D}(A)_T$ has Hodge filtration $\omega_{A/T}$. By Corollary 6.7 of [4] the crystals

$$\mathbb{D}(A) \cong \mathbb{D}(A_0)$$

are isomorphic since A is a deformation of A_0. Hence the Hodge filtration of $\mathbb{D}(A)_T$ necessarily restricts to the Hodge filtration $\omega_{A_0/T_0} = \omega_{A_{n-1}/T_0} \oplus \omega_{E/T_0}$ of

$$\mathbb{D}(A_0)_{T_0} \cong \mathbb{D}(A_{n-1})_{T_0} \oplus \mathbb{D}(E)_{T_0}.$$

Note that the projection of $\omega_{A/T}$ to each of the summands $\mathbb{D}(A_{n-1})_T$ and $\mathbb{D}(E)_T$ of $\mathbb{D}(A)_T$ has to restrict to ω_{A_{n-1}/T_0} and ω_{E/T_0}, respectively, but that since A_{n-1} is defined over Z there is a priori no Hodge filtration for $\mathbb{D}(A_{n-1})_T$.

We can work with either summand of

$$\mathbb{D}(A) \cong \mathbb{D}(A)^+ \oplus \mathbb{D}(A)^-$$

since they are dual to each other under the polarization. Both are locally free rank n modules which fit into the same kind of short exact sequences as $\mathbb{D}(A)$:

$$\omega^-_{A/T} \xrightarrow{f} \mathbb{D}(A)^-_T \longrightarrow \mathrm{Lie}^+_{A/T}.$$

Remember, that the ranks of $\omega^-_{A/T}$ and $\mathrm{Lie}^+_{A/T}$ were specified by the signature condition to 1 and $n-1$, respectively.

Without loss of generality we will stick to

$$\mathbb{D}(A)^-_T \cong \mathbb{D}(A_{n-1})^-_T \oplus \mathbb{D}(E)^-_T$$

and show that in fact the composition $(\mathrm{pr}_{A_{n-1}} \circ f)(\omega^-_{A/T}) \subseteq \mathbb{D}(A_{n-1})^-_T$ is a direct summand. As immediate consequence of Grothendieck-Messing theory there will then be a deformation of A_{n-1} to T which gives the desired T-point of Z.

Since

$$\mathbb{D}(E)^-_{T_0} \cong \mathrm{Lie}_{E/T_0}$$

the Hodge filtration of $\mathbb{D}(A_0)^-_{T_0} \cong \mathbb{D}(A_{n-1})^-_{T_0} \oplus \mathrm{Lie}_{E/T_0}$ is given by

$$\omega^-_{A_0/T_0} \cong \omega^-_{A_{n-1}/T_0}.$$

By our above discussion of the restriction of Hodge filtrations we then conclude that $(\mathrm{pr}_E \circ f)(\omega^-_{A/T})$ has to vanish mod I, where $I = \mathcal{I}(T)$ is the ideal defining T_0 in T. Equivalently, we can say that the map f is given by $1 \mapsto (a_1, \cdots, a_{n-1}, a_n)$ with $a_n \in I$.

Lemma 4.1 now shows that the map

$$\mathrm{pr}_{A_{n-1}} \circ f : \omega^-_{A/T} \longrightarrow \mathbb{D}(A_{n-1})^-_T$$

from the Hodge filtration to the summand $\mathbb{D}(A_{n-1})^-_T$ is a split injection.

□

As an immediate consequence of this theorem we find:

Corollary 4.3.

$$\mathcal{O}_{\widehat{\mathcal{M}}^{pd}_Z} \cong \mathcal{O}_Z \oplus \mathcal{I}.$$

□

4.2. **A canonical section of the normal bundle.** Consider the map $f : E \longrightarrow A$ defining the divisor Z. By naturality of the short exact sequence of the Dieudonné crystal f induces a diagram

$$\begin{tikzcd}
\omega^-_{A(p)} \arrow[r] \arrow[d, dashed] & \mathbb{D}(A(p))^- \arrow[r] \arrow[d, "f^*"'] & \mathrm{Lie}(A(p)^\vee)^- \arrow[d, dashed] \\
\omega^-_{E(p)} \arrow[r] & \mathbb{D}(E(p))^- \arrow[r] & \mathrm{Lie}(E(p)^\vee)^-.
\end{tikzcd}$$

By Grothendieck-Messing deformation theory the existence of the dashed maps is equivalent to a deformation of f to the corresponding nilpotent thickening. By maximality of the moduli stack Z the map f does not exist anywhere, but on Z itself and we obtain a non trivial map of $\mathcal{O}_{\widehat{M}^{pd}_Z}$-modules

$$s : \omega^-_{A(p)} \longrightarrow \mathbb{D}(A(p))^- \xrightarrow{f^*} \mathbb{D}(E(p))^- \longrightarrow \mathrm{Lie}_{E(p)}.$$

Note that thus

$$s \in \mathrm{Hom}_{\widehat{M}^{pd}_Z}\left(\omega^-_{A(p)}, \mathrm{Lie}_{E(p)}\right) \cong \mathrm{Hom}_{\widehat{M}^{pd}_Z}\left(\mathcal{O}_{\widehat{M}^{pd}_Z}, r^*(\omega^-_A \otimes \omega_E)^\vee\right)$$

vanishes to order one on Z and nowhere else. Hence

$$s \in \mathrm{Hom}_{\widehat{M}^{pd}_Z}\left(\mathcal{O}_{\widehat{M}^{pd}_Z}, r^*(\omega^-_A \otimes \omega_E)^\vee \otimes J\right).$$

Proposition 4.4.

$$\mathcal{O}_{\widehat{M}^{pd}_Z} \xrightarrow{\cong} r^*(\omega^-_A \otimes \omega_E)^\vee \otimes_{\mathcal{O}_{\widehat{M}^{pd}_Z}} J,$$

and hence

$$r^*(\omega^-_A \otimes \omega_E) \xrightarrow{\cong} J$$

as $\mathcal{O}_{\widehat{\mathcal{M}}_Z^{pd}}$-modules.

Proof. The map is given by $g \mapsto s \cdot g$ and since s is a global section of the locally trivial bundle $(\omega_A^- \otimes \omega_E)^\vee$ every other section vanishing on the divisor is a multiple of s. Now we have a surjective map of rank one $\mathcal{O}_{\widehat{M}_Z^{pd}}$-module sheaves and thus an isomorphism. $\square$

To simplify notation we may frequently drop the retraction r from the notation. Hence the $\mathcal{O}_Z$-module $\omega_A^- \otimes \omega_E$ has to be interpreted suitably.

Corollary 4.5. *For all pd-nilpotent thickenings* $\mathrm{Inf}^n_{\mathcal{D}}(Z)$ *we find*

$$D_{\mathcal{O}_{\widehat{\mathcal{M}}_Z}}(Z)/J^{[n+1]} \cong (\omega_A^- \otimes \omega_E)^\vee \otimes_{\mathcal{O}_{\widehat{M}_Z^{pd}}} J/JJ^{[n+1]}.$$

In particular, for odd primes

$$\mathcal{O}_Z \cong (\omega_A^- \otimes \omega_E)^\vee \otimes_{\mathcal{O}_{\widehat{M}_Z^{pd}}} J/J^{[2]},$$

and hence $J/J^{[2]}$ *is isomorphic to* $\omega_A^- \otimes \omega_E$ *as* $\mathcal{O}_Z$*-modules.*

Proof. Use the statement of the proposition, tensor with $\mathcal{O}_{\widehat{M}_Z^{pd}}/J^{[n+1]}$ over $\mathcal{O}_{\widehat{M}_Z^{pd}}$ and identify $\mathcal{O}_{\widehat{\mathcal{M}}_Z^{pd}}/J^{[n+1]}$ with $D_{\mathcal{O}_{\widehat{\mathcal{M}}_Z}}(Z)/J^{[n+1]}$. For odd primes $J^2 \cong J^{[2]}$, since $\frac{1}{2} \in \mathbb{Z}_p$. As $\mathcal{O}_Z \cong D_{\mathcal{O}_{\widehat{\mathcal{M}}_Z}}(Z)/J$ the claim follows. $\square$

We are now in the position to prove our first main result.

Theorem 4.6. *Let $p > 2$. The structure sheaf $\mathcal{O}_{\widehat{\mathcal{M}}_Z^{pd}}$ of the divided power completion is as $\mathcal{O}_Z$-algebra isomorphic to the sheaf of divided power series rings* $\mathcal{O}_Z \left\langle\left\langle \omega_A^- \otimes \omega_E \right\rangle\right\rangle$.

Proof. By Theorem 4.2 the divisor Z is a retract of $\widehat{\mathcal{M}}_Z^{pd}$ so we get a map

$$\mathcal{O}_Z \longrightarrow \mathcal{O}_{\widehat{\mathcal{M}}_Z^{pd}},$$

which in particular is the inclusion of a direct summand. Let $\mathcal{L}$ denote $J/J^{[2]}$, then by Proposition 4.4 and Corollary 4.5 the section s now provides a surjective map of $\mathcal{O}_Z$-algebras

$$\mathcal{O}_Z \left\langle\left\langle \mathcal{L} \right\rangle\right\rangle \longrightarrow \mathcal{O}_{\widehat{\mathcal{M}}_Z^{pd}}$$

that sends $\mathcal{L}$ to J considered as $\mathcal{O}_Z$-module via

$$\mathcal{L} \xrightarrow{adj} r_* r^* \mathcal{L} \cong \mathcal{L} \otimes_{\mathcal{O}_Z} \mathcal{O}_{\widehat{\mathcal{M}}_Z^{pd}} \xrightarrow{\cong} r_* J.$$

Since this map induces an isomorphism on the associated gradeds, it is itself an isomorphism of $\mathcal{O}_Z$-algebras. The second part of Corollary 4.5 now gives the claim. $\square$

The notation $\mathcal{O}_Z \left\langle\left\langle \mathcal{L} \right\rangle\right\rangle$ refers to the sheaf defined locally, where the invertible sheaf $\mathcal{L}$ is generated by a single generator, to be the sheaf of divided power series rings over $\mathcal{O}_Z$ on this single generator.

Corollary 4.7. *Let ω denote ω_A^-. At odd primes p we have,*

$$H^0(\widehat{\mathcal{M}}_{\mathcal{Z}}^{pd}, \omega^{\otimes l}) \cong H^0(\mathcal{Z}, \omega^{\otimes l} \left\langle\left\langle \omega \otimes \omega_E \right\rangle\right\rangle).$$

$\square$

4.3. **A remark on ordinary completions.** Surprisingly, we are able to prove that an existence result as in Theorem 4.2 also holds for the case of ordinary formal completion $\widehat{\mathcal{M}}_Z$. We will briefly explain the argument and why it is not of any use for our purposes.

Proposition 4.8. *Z is a retract of $\widehat{\mathcal{M}}_Z$. In other words, there is a canonical retraction*

$$\begin{tikzcd} Z \arrow[r] & \widehat{\mathcal{M}}_Z . \arrow[l, bend right, dashed] \end{tikzcd}$$

Proof. The strategy of proof is similar to the divided power case. We will find a deformation of A_{n-1} to $\widehat{\mathcal{M}}_Z$, but instead of Grothendieck-Messing deformation theory we use Serre-Tate theory.

As mentioned before via the polarization it suffices to work with either summand in the splitting of the p-divisible groups in question. We chose to work with the summand $E(u^c)$ which is purely étale.

It is well known that étale p-divisible groups (or dually multiplicative ones) lift uniquely through nilpotent thickenings. The basic argument is like follows.

Since étaleness is an open condition on the base we work locally and assume the base $\operatorname{Spec} R$ is affine. Then $E(u^c)$ is given as the colimit of affines $\operatorname{Spec} A_k$, for finite étale R-algebras A_k. If now $R_0 = R/I$ for some nilpotent ideal I, then A_k lifts uniquely and functorially to a finite étale R-algebra B_k . By functoriality the Hopf algebra structure on A_k lifts uniquely to a Hopf algebra structure on B_k and thus $\operatorname{Spec} B_k$ is the unique finite étale group scheme deforming $\operatorname{Spec} A_k$ to $\operatorname{Spec} R$.

By Grothendieck existence theorem there is thus a unique deformation of $E(u^c)$ to the whole ordinary formal completion $\widehat{\mathcal{M}}_Z$.

Note that deformations of étale p-divisible groups behave well as quotients, i.e. lift as quotients, whereas dually multiplicative groups behave well as subgroups. Hence, as A and thus $A(p)$ is defined on all of $\mathcal{M}$, we can consider there kernel

$$\ker (f^\vee)^{(n)} \subseteq A(u^c)^{(n)} \longrightarrow E(u^c)^{(n)}$$

of the map induced by the dual of the map f defining the divisor on n-th nilpotent order. This kernel is a p-divisible group defining a deformation of $A_{n-1}(u^c)$ to the n-th nilpotent order. Again by Grothendieck existence theorem we also get a deformation of $A_{n-1}(u^c)$ to the whole ordinary formal completion $\widehat{\mathcal{M}}_Z$ which by Serre-Tate deformation theory gives the desired retraction.

□

Having this retraction at hand one would like to proceed in proving the analog of Theorem 4.6. The same argument as before would work, provided the existence of the section s on the ordinary completion. Unfortunately, we believe that there might not be such a section. By the lack of divided power structures on the ideal I defining Z in $\mathcal{M}$ one can use crystals naively only up to define s up to $(p-1)$-st order.

One would like to argue that there should be a map

$$f^* \colon \mathbb{D}(A(u^c)) \longrightarrow \mathbb{D}(E(u^c))$$

and define s as before, but we are not aware of any such proof.

We remark though that this map can not come from a map

$$E(u^c)_{/\widehat{\mathcal{M}}_Z} \longrightarrow A(u^c)_{/\widehat{\mathcal{M}}_Z}$$

on the level of p-divisible groups. A map like this would pair with the lift

$$E(u)_{/\widehat{\mathcal{M}}_Z} \longrightarrow A(u)_{/\widehat{\mathcal{M}}_Z},$$

which exists (dually to the above argument) since $E(u)$ is multiplicative, to give a map

$$E(p) \longrightarrow A(p)$$

which by Serre-Tate theory is equivalent to a deformation of the map f defining Z to $\widehat{\mathcal{M}}_Z$. Contradiction.

5. An expansion

Lemma 5.1. *The map*

$$Z \longrightarrow \mathcal{M}$$

is surjective on π_0.

Proof. First, note that our stacks are noetherian and smooth, hence irreducible components coincide with connected components. Let us show that it is sufficient to check the claim over $\mathbb{C}$-points.

Since we inverted all ramified primes, there cannot be any connected component solely supported in characteristic p, i.e. there is always a point of characteristic zero. In addition, by flatness of $\mathcal{M}$ every geometric connected component in characteristic p lies in the closure of a geometric connected component in characteristic zero.

The existence of toroidal compactifications as provided by [10] for the moduli stack $\mathcal{M}_{(n-1,1)}$ and the fact that $\mathcal{M}_{(n-1,1)}$ is fiberwise dense in its toroidal compactification determines that there is only one such geometric connected component in characteristic p in each such closure. The key here is that properness combined with smoothness implies that the number of connected components is constant.

Note that the toroidal compactification $\mathcal{M}^\tau = \mathcal{M}^\tau_{(n-1,1)} \times_{\mathcal{O}_K} \mathcal{M}_{(1,0)}$ depends only on the toroidal compactification of $\mathcal{M}_{(n-1,1)}$ since $\mathcal{M}_{(1,0)}$ is proper. If we can show that the special fiber of every connected component of the divisor Z is non-empty, it is indeed sufficient to check the claim of the lemma on $\mathbb{C}$-points. As the number of geometric connected components does not change by considering a larger algebraically closed field.

By complex uniformization we can find a complex multiplication point x in every connected component of Z, i.e. a septuple $(A, i_A, \lambda, E, i_E, \lambda_0, f) \in Z(S)$ where both A and E have complex multiplication. Hence they in particular have potentially good reduction. So the theory of Neron models allows to extend the septuple to a discrete valuation ring with residue field in characteristic p. The reduction gives a mod p point in the connected component, which proves that the special fiber of every connected component is non-empty.

We are now in the position to check the claim on $\mathbb{C}$-points. Let us denote by $D(\Lambda)$ the space of negative 1-planes in the hermitian vector space $\Lambda \otimes \mathbb{Q}$. Then Proposition 3.1 and Proposition 3.5 of [8] use complex uniformization to provide the following presentations

$$\mathcal{M}(\mathbb{C}) \cong \coprod_{\Lambda} [\Gamma_\Lambda \backslash D(\Lambda)],$$

where the sum runs over all self-dual hermitian $\mathcal{O}_K$-lattices Λ of signature $(1, n-1)$, and

$$Z(\mathbb{C}) \cong \coprod_{[a]\in C(K)} \coprod_{\Lambda} \left[\left(\mathcal{O}_K^\times \times \Gamma_\Lambda\right) \Big\backslash \coprod_{\substack{x\in\Lambda \\ (x,x)=1}} D(\Lambda)_x \right]$$

where we additionally sum over all fractional ideals in the class group of K and $D(\Lambda)_x$ is the orthogonal complement of x in $D(\Lambda)$. Hence $\alpha_{\mathbb{C}}$ is a map

$$\coprod_{[a]\in C(K)} \coprod_{\Lambda} \left[\left(\mathcal{O}_K^\times \times \Gamma_\Lambda\right) \Big\backslash \coprod_{\substack{x\in L_a \\ (x,x)_a=1}} D(\Lambda)_x \right] \longrightarrow \coprod_{\Lambda} [\Gamma_\Lambda \backslash D(\Lambda)].$$

Now the claim is immediately clear, since $D(\Lambda)_x$ embeds into $D(\Lambda)$ by definition and the coset presentation on both sides is indexed over the self-dual lattices.

□

Lemma 5.2. *The map*

$$\mathcal{O}_{\widehat{\mathcal{M}}_Z} \longrightarrow \mathcal{O}_{\widehat{\mathcal{M}}_Z^{pd}}$$

of sheaves of (graded) rings is injective.

Proof. There are inclusions $\mathcal{O}_{\mathcal{M}} \overset{i}{\subseteq} D_{\mathcal{M}}(I)$ and $I \subseteq J$ by definition of the divided power envelope. The map in question is the inverse limit of a system of maps

$$\mathcal{O}_{\mathcal{M}}/I^n \longrightarrow D_{\mathcal{M}}(I)/J^{[n]},$$

all of which are injective, since the preimage of $J^{[n]}$ under i is precisely I^n by definition of J. Inverse limits are left exact.

□

Note that the map of the proposition is not an isomorphism of graded $\mathcal{O}_{\widehat{\mathcal{M}}_Z}$-algebras, as it is not surjective (except over the rationals). On graded pieces there are maps given by $x \mapsto x^{[n]}$ which provide an isomorphism of graded modules, but these are not compatible with the multiplications.

Theorem 5.3. *Let p be odd, L be an $\mathbb{Z}_p$-module and let us denote ω_A^- by just ω. The restriction-of-sections map*

$$\mathrm{H}^0\big(\mathcal{M}, \omega^{\otimes n} \otimes L\big) \longrightarrow \mathrm{H}^0\Big(\widehat{\mathcal{M}}_Z, \omega^{\otimes n} \otimes L\Big)$$

is injective.

Proof. By considering the ring of dual numbers of L we reduce to the case where L is an $\mathcal{O}_{K,u}$-*algebra.*

Let φ be an element in $H^0(\mathcal{M}_{/L}, \omega^{\otimes k})$ which restricts to zero on the formal completion. Then φ is still zero when restricted further to a formal neighborhood $\widehat{\mathcal{M}}_x$ of any geometric point $x \in Z$ (or more precisely a point x_i in each connected component of Z).

Locally around such points we can identify $\omega^{\otimes k}$ with the structure sheaf $\mathcal{O}_{\mathcal{M}}$ and thus regard φ as a section of the formal completion $\mathcal{O}_{\widehat{\mathcal{M}}_x}$ of the stalk $\mathcal{O}_{\mathcal{M}_{(x)}}$. For each connected component Krull's intersection theorem then ensures that the canonical completion map

$$\begin{tikzcd} \omega^{\otimes n}_{\mathcal{M}_{(x)}} = \omega^{\otimes n} \otimes_{\mathcal{M}} \mathcal{O}_{\mathcal{M}_{(x)}} \arrow[r, "compl"] \arrow[d, "\cong"'] & \omega^{\otimes n} \otimes_{\mathcal{M}} \mathcal{O}_{\widehat{\mathcal{M}}_x} = \omega^{\otimes n}_{\widehat{\mathcal{M}}_x} \arrow[d, "\cong"'] \\ \mathcal{O}_{\mathcal{M}_{(x)}} \arrow[r, "compl"] & \mathcal{O}_{\widehat{\mathcal{M}}_x} \end{tikzcd}$$

from the stalk of $\omega^{\otimes k}$ at the point x_i to its formal completion is injective. By definition of stalks there is thus a Zariski open set containing the formal neighborhood of x_i on which φ vanishes. Since the connected component is irreducible this open neighborhood is in particular dense.

Note that we could also work with the strict Henselization of the stalk to get an dense open in the étale topology.

Lemma 5.1 ensures we find a point x_i in every connected component of $\mathcal{M}$, hence the zero locus of φ is dense open in every connected component of $\mathcal{M}$. Since by definition the zero locus of a section is locally closed, hence closed, φ vanishes on all of $\mathcal{M}$.

□

Corollary 5.4. *Let ω denote ω_A^-. At odd primes there is an injective ring homomorphism*

$$T\colon\ \mathrm{H}^0\big(\mathcal{M}, \omega^{\otimes k}\big) \longrightarrow \mathrm{H}^0\big(Z, \omega^{\otimes k}\langle\langle \omega \otimes \omega_E \rangle\rangle\big)\,.$$

Proof. We combine Theorem 5.3

$$\mathrm{H}^0\big(\mathcal{M}, \omega^{\otimes k}\big) \longrightarrow \mathrm{H}^0\left(\widehat{\mathcal{M}}_Z, \omega^{\otimes k}\right)$$

with Lemma 5.2

$$\mathrm{H}^0\left(\widehat{\mathcal{M}}_Z, \omega^{\otimes k}\right) \longrightarrow \mathrm{H}^0\left(\widehat{\mathcal{M}}^{pd}_Z, \omega^{\otimes k}\right)$$

and Corollary 4.7

$$\mathrm{H}^0\left(\widehat{\mathcal{M}}^{pd}_Z, \omega^{\otimes k}\right) \cong \mathrm{H}^0\big(Z, \omega^{\otimes k}\langle\langle \omega \otimes \omega_E \rangle\rangle\big)$$

and denote the composition by T.

□

6. An expansion principle

In Corollary 5.4 we put together result of the earlier sections to produce an expansion map. As an immediate consequence of the construction we deduce that this expansion map behaves well with coefficients L of characteristic 0.

Proposition 6.1. *Let L be a $\mathbb{Z}_p$-algebra of characteristic 0, then the expansion map of forms with coefficients in L*

$$T_L\colon\ \mathrm{H}^0\big(\mathcal{M}, \omega^{\otimes k} \otimes_{\mathbb{Z}_p} L\big) \longrightarrow \mathrm{H}^0\big(Z, \omega^{\otimes k}\langle\langle \omega \otimes \omega_E \rangle\rangle \otimes_{\mathbb{Z}_p} L\big)\,.$$

remains injective. □

Remark 6.2. However, we must note that things do not behave well in characteristic p. If say x is an automorphic form with coefficients in an $\mathbb{Z}_p$-algebra L of characteristic p, then x will still be sent to x under the map

$$\mathrm{H}^0\left(\widehat{\mathcal{M}}_{Z,L}, \omega^{\otimes k}\right) \to \mathrm{H}^0\left(\widehat{\mathcal{M}}^{pd}_{Z,L}, \omega^{\otimes k}\right)$$

induced by the map of Lemma 5.2, but since $x^p = p! \cdot x^{[p]} \equiv 0 \mod p$ this map, and thus the expansion map T_L, is no longer injective. The same happens for $\mathbb{Z}_p$-modules with torsion.

The statements of Theorem 4.6, i.e. the injectivity of the restriction map to the ordinary formal completion, and Theorem 5.3, i.e. the interpretation of forms on the divided power completion as power series, remain correct in characteristic p.

In the end, we can derive only a faint shadow of an expansion principle, since as the remark indicates we can't deal with torsion in the quotient of $\mathbb{Z}_p$-algebras.

Proposition 6.3. *Let $L_1 \subseteq L_2$ be $\mathbb{Z}_p$-algebras of characteristic 0, with torsion free quotient. Then the diagram*

$$\begin{array}{ccc} \mathrm{H}^0\big(\mathcal{M}_{/L_1}, \omega^{\otimes k}\big) & \xrightarrow{T_{L_1}} & \mathrm{H}^0\big(Z_{/L_1}, \omega^{\otimes k}\langle\langle \omega \otimes \omega_E \rangle\rangle\big) \\ \downarrow & & \downarrow \\ \mathrm{H}^0\big(\mathcal{M}_{/L_2}, \omega^{\otimes k}\big) & \xrightarrow{T_{L_2}} & \mathrm{H}^0\big(Z_{/L_2}, \omega^{\otimes k}\langle\langle \omega \otimes \omega_E \rangle\rangle\big) \end{array}$$

is a pullback.

Proof. Applying the functor $\mathrm{H}^0\big(\mathcal{M}, \omega^{\otimes n} \otimes_{\mathcal{O}_{k,u}} -\big)$ to

$$0 \longrightarrow L_1 \longrightarrow L_2 \longrightarrow L_2/L_1 \longrightarrow 0$$

and considering the expansion map of Proposition 6.1 at each step yields the following commutative diagram of exact sequences

$$\begin{array}{ccccc} \mathrm{H}^0\big(\mathcal{M}_{/L_1}, \omega^{\otimes k}\big) & \longrightarrow & \mathrm{H}^0\big(\mathcal{M}_{/L_2}, \omega^{\otimes k}\big) & \longrightarrow & \mathrm{H}^0\big(\mathcal{M}, \omega^{\otimes k} \otimes L_2/L_1\big) \to \cdots \\ \downarrow{\scriptstyle T_{L_1}} & & \downarrow{\scriptstyle T_{L_2}} & & \downarrow{\scriptstyle T_{L_2/L_1}} \\ \mathrm{H}^0\big(Z_{/L_1}, \omega^{\otimes k}\langle\langle \omega \otimes \omega_E \rangle\rangle\big) & \longrightarrow & \mathrm{H}^0\big(Z_{/L_2}, \omega^{\otimes k}\langle\langle \omega \otimes \omega_E \rangle\rangle\big) & \longrightarrow & \mathrm{H}^0\big(Z, \omega^{\otimes k}\langle\langle \omega \otimes \omega_E \rangle\rangle \otimes L_2/L_1\big). \end{array}$$

Let φ be an element of $\mathrm{H}^0\big(\mathcal{M}_{/L_2}, \omega^{\otimes k}\big)$. Remember that all vertical maps are injective, so a form φ vanishes if and only if all coefficients in the expansion $T_{L_2}(\varphi) = \sum \alpha_{n+k}$ vanish. Hence if the image of $T_{L_2}(\varphi)$ vanishes under the lower right horizontal map, then $T_{L_2}(\varphi)$ is the image of a form $\theta \in \mathrm{H}^0\big(Z_{/L_1}, \omega^{\otimes k}\langle\langle \omega \otimes \omega_E \rangle\rangle\big)$ and the image of φ under the upper right horizontal map is zero. By exactness φ then comes from a form $\varphi' \in \mathrm{H}^0\big(Z_{/L_1}, \omega^{\otimes k}\big)$ with $T_{L_1}(\varphi') = \theta$. $\square$

We remark again that this result does not what it was supposed to do. In particular, it can't be used to check integrality, since $\mathbb{Q}_p/\mathbb{Z}_p$ is the prototype of a torsion module. We outline this merely useless expansion principle anyways, to emphasize that the only thing that the only thing that debilitates this theorem is the use of divided powers. If one could evade those, then the same proofs work.

References

[1] Leovigildo Alonso Tarrío, Ana Jeremías López, and Marta Pérez Rodríguez. Infinitesimal lifting and Jacobi criterion for smoothness on formal schemes. April 2006.

[2] Paul Balmer. Spectra, spectra, spectra—tensor triangular spectra versus Zariski spectra of endomorphism rings. *Algebr. Geom. Topol.*, 10(3):1521–1563, 2010.

[3] Mark Behrens and Tyler Lawson. Topological automorphic forms. *Mem. Amer. Math. Soc.*, 204(958):xxiv+141, 2010.

[4] Pierre Berthelot and Arthur Ogus. *Notes on crystalline cohomology*. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1978.

[5] Paul G. Goerss. Topological modular forms [after Hopkins, Miller and Lurie]. *Astérisque*, (332):Exp. No. 1005, viii, 221–255, 2010. Séminaire Bourbaki. Volume 2008/2009. Exposés 997–1011.

[6] Benjamin Howard. Complex multiplication cycles and Kudla-Rapoport divisors, II. *Amer. J. Math.*, 137(3):639–698, 2015.

[7] N. Krämer. Local models for ramified unitary groups. *Abh. Math. Sem. Univ. Hamburg*, 73:67–80, 2003.

[8] Stephen Kudla and Michael Rapoport. Special cycles on unitary Shimura varieties II. Global theory. December 2012.

[9] Stephen Kudla, Michael Rapoport, and Tonghai Yang. On the derivative of an Eisenstein series of weight one. *Internat. Math. Res. Notices*, (7):347–385, 1999.

[10] Kai-Wen Lan. *Arithmetic compactifications of PEL-type Shimura varieties*, volume 36 of *London Mathematical Society Monographs Series*. Princeton University Press, Princeton, NJ, 2013.

[11] Georgios Pappas. On the arithmetic moduli schemes of PEL Shimura varieties. *J. Algebraic Geom.*, 9(3):577–605, 2000.