Non-Overlapping Domain Decomposition Methods Interpreted as Multiplicative Subspace Correction Algorithms

R. Verfürth

Fakultät für Mathematik, Ruhr-Universität Bochum, D-44780 Bochum, Germany

E-mail address: rv@silly.num1.ruhr-uni-bochum.de

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Summary: We interprete non-overlapping domain decomposition methods as multiplicative subspace correction algorithms in the framework of Xu [8]. This allows us to estimate the effects of the perturbation which is created by an inexact solution of the problems on the subdomains that must be solved in each iteration. The general results are applied to finite element discretizations of the Poisson and Stokes equations.

Key words: non-overlapping domain decomposition methods; multiplicative subspace correction methods; finite element methods; parallelization.

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1. Introduction

Non-overlapping domain decomposition methods are well-established and efficient algorithms for the solution of the large algebraic systems arising from finite element of finite difference discretizations of elliptic partial differential equations. They are well adapted to modern parallel computer architectures since they split the original problem into independent problems on the subdomains, which can be solved completely in parallel, and a lower dimensional problem on the interface of the subdomains. The latter usually is solved by some preconditioned conjugate gradient algorithm (cf e.g. [1, 2, 3, 7]). The evaluation of the operator associated with the interface problem and of the preconditioner typically involves some kind of "harmonic" extension of data given on the interface to the whole domain. This extension amounts in the solution - separate on each subdomain - of a problem similar to the original one. Thus the main task of the algorithm consists in solving independent problems on the subdomains. This is usually done only approximately. The approximate solution of these subproblems of course corresponds to a perturbation of the original algorithm and influences its convergence speed. It is the aim of this paper to estimate the effects of this perturbation.

We achieve our goal by interpreting the domain decomposition method as a multiplicative subspace correction algorithm in the framework of Xu [8]. The domain

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decomposition method with an exact solution of the problems on the subdomains then corresponds to an orthogonal splitting of the function space, which is associated with the underlying variational problem, into a subspace, which consists of functions vanishing on the interface, and its complement the elements of which are completely determined by their values on the interface. The inexact solution of the subproblems only affects the second component of this splitting. The orthogonality is thus destroyed. The angle between the new subspaces, however, can be controlled by the error which is admitted in the solution of the subproblems. Moreover, the functions in the second component of the resulting splitting are still completely determined by their values on the interface. The splitting of the function space introduces a decomposition of the differential operator. One component corresponds to the subspace of functions vanishing on the interface, the other one to the complementary subspace. Thus the first component is not affected by an inexact solution of the subproblems. The second one on the other hand is canonically associated with a Poincaré-Steklov operator acting on a suitable trace space on the interface. This operator is affected by an inexact solution of the subproblems. Its perturbation can completely be controlled by the error which is commited in the solution of the subproblems. Therefore, any good preconditioner for the problem corresponding to the ideal splitting will still be a reasonable preconditioner for the perturbed one. Moreover, its condition number can be controlled by the accuracy invested into the solution of the subproblems.

In order to clarify our point of view we consider in section 2 the Poisson equation as a simple model problem and interprete the corresponding non-overlapping domain decomposition as an alternating projection method. In order to be independent of a particular discretization and to highlight the essential points, we do this within the infinite dimensional variational setting. In section 3 we shortly review the multiplicative subspace correction method of Xu [8]. Its convergence behaviour can be completely described by four quantities. Two of them are related to the angle between the subspaces and are therefore directly influenced by a perturbation of the splitting. The other two are related to the preconditioning of the operators and are therefore only indirectly influenced by a perturbation of the splitting. In section 4, which is the central part of this paper, we give an abstract perturbation result which is inspired by the example of section 2 and which gives control of the relevant quantities of section 3. In section 5 we apply the abstract results to two concrete examples: the finite element discretization of the Poisson equation and certain mixed finite element discretizations of the Stokes equations. The first example immediately carries over to all those scalar linear elliptic equations of second order which are the Euler-Lagrange equations of a convex quadratic functional on an affine subspace of $H^1(\Omega)$.

2. Non-Overlapping Domain Decomposition Methods

In order to motivate the point of view of the following analysis, consider the simple model problem

$$\begin{aligned} -\Delta u &= f \quad \text{in}\,\Omega \\ u &= 0 \quad \text{on}\,\Gamma. \end{aligned} \tag{2.1}$$

Here, $\Omega \subset \mathbb{R}^d$, $d \geq 2$, is an open, bounded, and connected domain with Lipschitz boundary Γ. Problem (2.1) is interpreted in the usual weak sense.

Split Ω into $N \geq 2$ pairwise disjoint open, bounded, and connected subdomains Ω_i, $1 \leq i \leq N$, each having a Lipschitz boundary such that

$$\bar{\Omega} = \bigcup_{1 \leq i \leq N} \bar{\Omega}_i.$$

Set

$$\omega := \bigcup_{1 \leq i \leq N} \Omega_i \quad , \quad \Sigma := \Omega \backslash \omega.$$

Σ is the interface of the subdomains. To simplify the exposition, we consider in this section only the analytical problem and postpone its finite element discretization to the following sections. Consequently we will work with $H^1(\Omega)$-functions. Since these do not admit well-defined point values, we will assume that Σ is the disjoint union of smooth hyperplanes (see Fig. 1). In the following sections, which deal with finite element spaces, we may allow for the general situation where Σ may have cross-points (see Fig. 2).

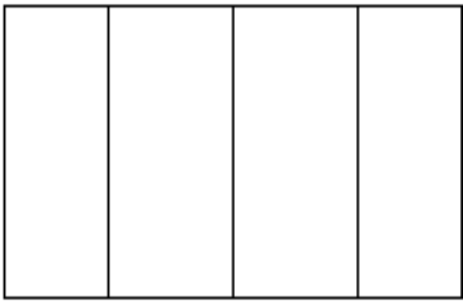

Fig. 1: Decomposition into 4 subdomains without cross-points

Fig. 2: Decomposition into 4 subdomains with cross-points

Let n be a unit normal to Σ. For a given function v having suitable regularity properties denote by $J(v)$ the jump of v across Σ in the direction n.

It is well-known (cf. e.g. [4; §I.4]) that u is a weak solution of (2.1) if and only if its trace φ on Σ solves

$$T\varphi = g. \tag{2.2}$$

Here, g is given by

$$g := -J(\partial_n u_\omega)$$

and u_ω is the unique weak solution of

$$\begin{aligned} -\Delta u_\omega &= f \quad \text{in } \omega \\ u_\omega &= 0 \quad \text{on } \partial\omega. \end{aligned} \tag{2.3}$$

Note, that (2.3) consists of N independent Poisson equations with homogeneous Dirichlet boundary conditions on the subdomains Ω_i, $1 \le i \le N$. The Poincaré-Steklov operator T is given by

$$T\varphi := J(\partial_n(E\varphi))$$

where $E\varphi$ denotes the harmonic extension of φ. $E\varphi$ is the unique weak solution of

$$\begin{aligned} -\Delta(E\varphi) &= 0 \quad \text{in } \omega \\ E\varphi &= 0 \quad \text{on } \Gamma \\ E\varphi &= \varphi \quad \text{on } \Sigma. \end{aligned} \tag{2.4}$$

Note, that (2.4) consists of N independent Poisson equations in the subdomains Ω_i, $1 \le i \le N$, with non-homogeneous Dirichlet boundary conditions on the interface Σ.

A non-overlapping domain decomposition method for (2.1) now consists of the following steps:

(1) Solve (2.3) and compute g.
(2) Solve (2.2).
(3) Compute $E\varphi$ by solving (2.4).
(4) Set $u := u_\omega + E\varphi$.

In pratice this algorithm is of course performed on the discrete level, i.e. problems (2.1)–(2.4) must be replaced by corresponding finite element or finite difference approximations. Having in mind this discrete realization, the main difficulty of the above algorithm lies in the solution of (the discrete counterparts of) problems (2.2)–(2.4). In practice this will be done only approximately, often using an iterative solver. In order to better unterstand the influence of the errors which are introduced by the approximate solution of (2.2)–(2.4), we will interprete the above algorithm as an alternating projection method.

To this end, set

$$V := H_0^1(\Omega) := \{v \in L^2(\Omega) : \nabla v \in L^2(\Omega)^d, v = 0 \,\text{on}\, \Gamma\}$$

and

$$V_1 := H_0^1(\omega) := \{v \in V : v = 0 \,\text{on}\, \Sigma\}.$$

Note that

$$V_1 = \bigoplus_{1 \le i \le N} H_0^1(\Omega_i)$$

where the decomposition is orthogonal with respect to the standard H^1-scalar product. Let V_2 be the orthogonal complement of V_1 in V with respect to the H^1-scalar product. Obviously, step (1) above is equivalent to projecting the solution u of (2.1) onto V_1.

We claim that steps (2) and (3) are equivalent to projecting u onto V_2. To see this, note that thanks to our conditions on Σ the trace operator is an isomorphism of V onto $W := H_{00}^{1/2}(\Sigma)$. Here, $H_{00}^{1/2}(\Sigma)$ is the interpolation space halfway between $H_0^1(\Sigma)$ and $L^2(\Sigma)$ and consists of all functions in $H^{1/2}(\Sigma)$ which have an appropriate decay close to $\partial\Sigma$ [5; p. 66]. Let φ be the solution of (2.2) and consider an arbitrary $\psi \in H_{00}^{1/2}(\Sigma)$. We then have

$$\begin{aligned}
\int_\Omega \nabla(E\varphi)\nabla(E\psi) &= \sum_{i=1}^N \int_{\Omega_i} \nabla(E\varphi)\nabla(E\psi) \\
&= \sum_{i=1}^N \{-\int_{\Omega_i} \Delta(E\varphi)E\psi + \int_{\partial\Omega_i} \partial_n(E\varphi)E\psi\} \\
&= \int_\Sigma J(\partial_n E\varphi)\psi \\
&= \int_\Sigma (T\varphi)\psi
\end{aligned}$$

and

$$\begin{aligned}
\int_\Omega \nabla u \nabla(E\psi) &= \int_\Omega fE\psi \\
&= \sum_{i=1}^N \int_{\Omega_i} (-\Delta u_\omega)E\psi \\
&= \sum_{i=1}^N \{-\int_{\Omega_i} u_\omega \Delta(E\psi) - \int_{\partial\Omega_i} \partial_n u_\omega E\psi + \int_{\partial\Omega_i} u_\omega \partial_n(E\psi)\} \\
&= \int_\Sigma J(\partial_n u_\omega)\psi \\
&= \int_\Sigma g\psi.
\end{aligned}$$

This proves the claimed equivalence. (The above calculation - as it stands - makes sense only for sufficiently smooth functions. All steps, however, can correctly be interpreted in the sense of traces and duality pairings.)

The interpretation of the non-overlapping domain decomposition algorithm as an alternating projection method clearly shows the effects of an approximate solution of problems (2.2)–(2.4). An approximate solution of (2.4) corresponds to a perturbation of E and thus of T and of V_2. In particular this perturbation will destroy the orthogonality of V_1 and V_2. However, if the perturbation of E is sufficiently small, the perturbed space V_2 will be nearly orthogonal to V_1. Moreover, any good preconditioner for T will still be a reasonable preconditioner for the perturbation of T. An approximate solution of (2.3), on the other hand, does not influence the decomposition of V and may be viewed as a preconditioning of the differential operator restricted to V_1.

3. Multiplicative Subspace Correction Methods

Let V be a finite dimensional Hilbert-space with scalar product $(.,.)$. In practice, $(.,.)$ will correspond to the L^2-scalar product or an equivalent one which may e.g. be obtained by mass-lumping. On V we consider a symmetric, positive definite operator A. A introduces the energy norm

$$\|u\|_1 := (Au, u)^{1/2} \quad \forall u \in V.$$

Given $f \in V$ we want to solve the problem

$$Au = f. \tag{3.1}$$

Let V_1, V_2 be two subspaces of V such that $V = V_1 + V_2$, i.e., to each $v \in V$ there exists at least one pair $(v_1, v_2) \in V_1 \times V_2$ such that $v = v_1 + v_2$. Denote by $Q_i : V \to V_i$, $i = 1, 2$, the projection of V onto V_i with respect to $(.,.)$. $A_i : V_i \to V_i, i = 1, 2$, is the restriction of A on V_i defined by

$$(A_i u, v) := (Au, v) \quad \forall u, v \in V_i, i = 1, 2.$$

Note, that A_1, A_2 are symmetric, positive definite operators on V_1 and V_2. Finally, denote by $R_i : V_i \to V_i, i = 1, 2$, symmetric, positive definite operators which approximate A_i^{-1}. The multiplicative subspace correction algorithm is then given by:

3.1 Algorithm. *Given $u_0 \in V$. For $n = 0, 1, ...$ and $i = 1, 2$ compute*

$$u_{n+i/2} := u_{n+(i-1)/2} + R_i Q_i (f - A u_{n+(i-1)/2}).$$

3.2 Remark. Usually Algorithm 3.1 is formulated with more than two subspaces. For our purposes, however, the above formulation is more practical. In the application to non-overlapping domain decomposition methods, the space V_1 further splits into orthogonal subspaces. This splitting of V_1, however, is not affected by the perturbation argument which will only influence the space V_2. □

The convergence analysis of Algorithm 3.1 involves four quantities λ, Λ, K_0, and K_1 which we will now explain. Set

$$\lambda := \min_{i=1,2} \lambda_{\min}(R_i A_i) \quad \text{and} \quad \Lambda := \max_{i=1,2} \lambda_{\max}(R_i A_i). \tag{3.2}$$

Without less of generality we may assume that R_1, R_2 are scaled such that

$$0 < \lambda \leq \Lambda < 2$$

holds. Note, that $R_i = A_i^{-1}, i = 1, 2$, if and only if $\lambda = \Lambda = 1$.

Since the mapping

$$(v_1, v_2) \in V_1 \times V_2 \longrightarrow v_1 + v_2 \in V$$

is surjective, the open mapping theorem implies that there is a constant K_0 with the following property: for each $u \in V$ there exist $v_i \in V_i$, $i = 1, 2$, such that

$$u = v_1 + v_2 \quad \text{and} \quad \{\sum_{i=1}^{2} \|v_i\|_1^2\}^{1/2} \leq K_0 \|u\|_1. \tag{3.3}$$

The Cauchy-Schwarz inequality on the other hand implies that there is a constant K_1 such that

$$\sum_{1 \leq i,j \leq 2} (A v_i, w_j) \leq K_1 \{\sum_{i=1}^{2} \|v_i\|_1^2\}^{1/2} \{\sum_{j=1}^{2} \|w_j\|_1^2\}^{1/2} \tag{3.4}$$

holds for all $v_i, w_i \in V_i, i = 1, 2$. A trivial estimate of course is $K_1 \leq 2$.
Note that $K_0 = K_1 = 1$ if V_1 and V_2 are A-orthogonal.

The following convergence result is proven in [7; Thm. 4.4].

3.3 Theorem. *Let u be the solution of (3.1). The iterates of Algorithm 3.1 satisfy*

$$\|u - u_{n+1}\|_1^2 \leq \big[1 - (\frac{2}{\Lambda} - 1)(\frac{\lambda}{\Lambda K_0 K_1})^2\big] \|u - u_n\|_1^2 \quad \forall n \geq 0. \tag{3.5}$$

3.4 Remark. Xu [7; Thm. 4.4] gives a stronger estimate for the convergence rate of Algorithm 3.1 than the one of Theorem 3.3. Estimate (3.5), however, is better suited for our purposes, since it clearly separates the effects of the splitting, which are expressed by K_0 and K_1, and of the preconditioning by R_1 and R_2, which are measured by λ and Λ. □

4. An Abstract Setting

We retain the notations of the previous section. Now, we will consider particular splittings $V = V_1 + V_2$. To this end let W be another finite dimensional Hilbert-space with scalar product $< ., . >$. We assume that V and W are coupled by a continuous and surjective trace operator $\gamma : V \to W$. Within the framework of the introductory example V and W correspond to finite dimensional approximations of $H_0^1(\Omega)$ and $H_{00}^{1/2}(\Sigma)$. γ is the standard trace operator which associates with a function defined on Ω its restriction to Σ.

Set

$$U_1 := \ker(\gamma) = \{v \in V : \gamma(v) = 0\}$$

and

$$U_2 := U_1^{\perp_A} = \{v \in V : (Av, w) = 0 \quad \forall w \in U_1\}.$$

Denote by $E : W \to V$ the maximal right inverse of γ which is defined by

$$E\varphi = \operatorname*{argmin}_{v \in \gamma^{-1}(\varphi)} \|v\|_1^2 \quad \forall \varphi \in W.$$

Another characterization of $E\varphi$, $\varphi \in W$, is given by

$$\gamma(E\varphi) = \varphi \quad \text{and} \quad (A(E\varphi), v) = 0 \quad \forall v \in U_1. \tag{4.1}$$

Within the framework of the introductory example E corresponds to the harmonic extension of functions on Σ.

The characterization (4.1) implies

$$U_2 = E(W).$$

Thus, we have $K_0 = K_1 = 1$ when using the splitting $V_i = U_i$, $i = 1, 2$.

In practice, the action $E\varphi$ is evaluated only approximately. We take this into account, by considering an approximation $E_\varepsilon : W \to V$ of E which satisfies for all $\varphi \in W$

$$\|E\varphi - E_\varepsilon\varphi\|_1 \leq \varepsilon\|E\varphi\|_1 \quad \text{and} \quad \gamma(E_\varepsilon\varphi) = \varphi. \tag{4.2}$$

Here, $0 < \varepsilon < \frac{1}{3}$ is arbitrary but fixed. In practice, $E_\varepsilon\varphi$ will often be the result of an iterative process applied to the linear system (4.1) with starting value 0 (see section 5). Now, we set

$$V_1 := U_1 \quad , \quad V_2 := E_\varepsilon(W). \tag{4.3}$$

The following theorem shows the influence of the replacement of $E(W)$ by $E_\varepsilon(W)$ on the constants K_0 and K_1.

4.1 Theorem. *The splitting (4.3) fulfills equations (3.3) and (3.4) with*

$$K_0 \leq \sqrt{\frac{1-\varepsilon}{1-2\varepsilon}}$$

and

$$K_1 \leq \frac{1}{1-\varepsilon}.$$

Proof. We first note that inequality (4.2) implies for all $\varphi \in W$

$$|\|E\varphi\|_1 - \|E_\varepsilon\varphi\|_1| \leq \varepsilon\|E\varphi\|_1$$

and thus

$$(1-\varepsilon)\|E\varphi\|_1 \leq \|E_\varepsilon\varphi\|_1 \leq (1+\varepsilon)\|E\varphi\|_1 \quad \forall\varphi \in W. \tag{4.4}$$

Next, consider arbitrary functions $v_1 \in V_1$ and $v_2 = E_\varepsilon\varphi \in V_2$. Inequality (4.4) and the A-orthogonality of V_1 and $E(W)$ imply

$$\begin{aligned}(Av_1, v_2) &= (Av_1, E_\varepsilon\varphi - E\varphi) \\ &\leq \|v_1\|_1\|E_\varepsilon\varphi - E\varphi\|_1 \\ &\leq \varepsilon\|v_1\|_1\|E\varphi\|_1 \\ &\leq \frac{\varepsilon}{1-\varepsilon}\|v_1\|_1\|v_2\|_1 \,.\end{aligned} \tag{4.5}$$

In order to prove (3.3), consider an arbitrary $v \in V$. Set

$$\varphi := \gamma(v) \quad , \quad v_2 := E_\varepsilon\varphi \in V_2 \quad , \quad v_1 := v - v_2 \in V_1.$$

Inequality (4.5) then yields

$$\begin{aligned}\|v\|_1^2 &= \|v_1\|_1^2 + \|v_2\|_1^2 + 2(Av_1, v_2)\\ &\geq \|v_1\|_1^2 + \|v_2\|_1^2 - \frac{2\varepsilon}{1-\varepsilon}\|v_1\|_1\|v_2\|_1\\ &\geq [1 - \frac{\varepsilon}{1-\varepsilon}][\|v_1\|_1^2 + \|v_2\|_1^2].\end{aligned}$$

This proves the first part of the assertion.
In order to prove (3.4) consider $v_i, w_i \in V_i, i = 1, 2$, with

$$\|v_1\|_1^2 + \|v_2\|_1^2 = \|w_1\|_1^2 + \|w_2\|_1^2 = 1.$$

Inequality (4.5) then yields

$$\begin{aligned}\sum_{1\leq i,j\leq 2}(Av_i, w_j) \leq& \|v_1\|_1^2\|w_1\|_1 + \frac{\varepsilon}{1-\varepsilon}\|v_1\|_1\|w_2\|_1\\ &+ \frac{\varepsilon}{1-\varepsilon}\|v_2\|_1\|w_1\|_1 + \|v_2\|_1\|w_2\|_1\\ \leq& \frac{1}{2}[\|v_1\|_1^2 + \|w_1\|_1^2 + \|v_2\|_1^2 + \|w_2\|_1^2]\\ &+ \frac{1}{2}\frac{\varepsilon}{1-\varepsilon}[\|v_1\|_1^2 + \|w_2\|_1^2 + \|v_2\|_1^2 + \|w_1\|_1^2]\\ =& 1 + \frac{\varepsilon}{1-\varepsilon}\\ =& \frac{1}{1-\varepsilon}.\end{aligned}$$

Together with an homogeneity argument, this proves the second assertion. □

Next, we investigate the influence of the replacement of $E(W)$ by $E_\varepsilon(W)$ on the restriction A_2 of A onto the corresponding subspaces. To this end we note that there are two isomorphisms i and i_ε which associate with any linear operator S of W into W a linear operator $i(S)$ of $E(W)$ into $E(W)$ and a linear operator $i_\varepsilon(S)$ of $E_\varepsilon(W)$ into $E_\varepsilon(W)$. These isomorphisms are defined by

$$\begin{aligned}(i(S)E\varphi, E\psi) =& < S\varphi, \psi >\\ =& (i_\varepsilon(S)E_\varepsilon\varphi, E_\varepsilon\psi) \quad \forall\varphi, \psi \in W, S \in \mathcal{L}(W, W).\end{aligned} \tag{4.6}$$

Put

$$T := i^{-1}(A_{|E(W)}) \quad , \quad T_\varepsilon := i_\varepsilon^{-1}(A_{|E_\varepsilon(W)}).$$

For later use we note that the identities

$$< T\varphi, \psi >= (AE\varphi, E\psi) \quad , \quad < T_\varepsilon\varphi, \psi >= (AE_\varepsilon\varphi, E_\varepsilon\psi) \tag{4.7}$$

hold for all $\varphi, \psi \in W$. In particular, T and T_ε are symmetric and positive definite. The identity (4.6) shows that a linear operator $R : E(W) \to E(W)$ is a good preconditioner for $A_{|E(W)}$ if and only if $i^{-1}(R) : W \to W$ is a good preconditioner for T. Moreover, the operator $R_\varepsilon := i_\varepsilon \circ i^{-1}(R)$, which is induced by R, then is a good preconditioner for $A_{|E_\varepsilon(W)}$ provided the condition number of $T^{-1}T_\varepsilon$ is close to 1. Thus the question of how the replacement of $E(W)$ by $E_\varepsilon(W)$ influences the restriction of A onto the corresponding subspaces is reduced to the investigation of the spectrum of T_ε relative to T.

4.2 Theorem. *The estimate*

$$|\frac{< T\varphi - T_\varepsilon\varphi, \varphi >}{< T\varphi, \varphi >}| \leq \varepsilon(2+\varepsilon) \tag{4.8}$$

holds for all $\varphi \in W$. In particular the condition number of $T^{-1}T_\varepsilon$ is not greater than $(1+3\varepsilon)/(1-3\varepsilon)$.

Proof. Consider an arbitrary $\varphi \in W$. Equation (4.7) implies

$$\|E\varphi\|_1^2 = (AE\varphi, E\varphi) =< T\varphi, \varphi > . \tag{4.9}$$

Together with inequalities (4.2) and (4.5) and equations (4.7) and (4.9) this yields

$$\begin{aligned} | < T\varphi - T_\varepsilon\varphi, \varphi > | &= |(AE\varphi, E\varphi) - (AE_\varepsilon\varphi, E_\varepsilon\varphi)| \\ &= |(A(E\varphi - E_\varepsilon\varphi), E\varphi) + (AE_\varepsilon\varphi, (E\varphi - E_\varepsilon\varphi))| \\ &\leq [\|E\varphi\|_1 + \|E_\varepsilon\varphi\|_1]\|E\varphi - E_\varepsilon\varphi\|_1 \\ &\leq (2+\varepsilon)\varepsilon\|E\varphi\|_1^2 \\ &= (2+\varepsilon)\varepsilon < T\varphi, \varphi > . \end{aligned}$$

This proves estimate (4.8).

Inequality (4.8) implies that the spectrum of $T^{-1}T_\varepsilon$ is contained in

$$[1-\varepsilon(2+\varepsilon), 1+\varepsilon(2+\varepsilon)] \subset [1-3\varepsilon, 1+3\varepsilon].$$

This establishes the bound on the condition number. □

5. Applications

In this section we apply the abstract results of the previous section to finite element approximations of some partial differential equations. To this end, let $\mathcal{T}$ be a partition of Ω into d-simplices or d-cubes which satisfies the usual admissibility conditions for finite element partitions. We assume that $\mathcal{T}$ is consistent with the domain decomposition of Ω, i.e., the interface Σ is the union of element boundaries. Equivalently, $\mathcal{T}$

induces on each subdomain an admissible finite element partition into d-simplices or d-cubes. Note, that this assumption in particular implies that Σ is polyhedral. Denote by $\mathcal{S}$ the partition of Σ into $(d-1)$-simplices or $(d-1)$-cubes which is induced by $\mathcal{T}$.

The Poisson equation

As a first example, we consider problem (2.1). Let $V \subset H_0^1(\Omega)$ be a Lagrangian finite element space corresponding to $\mathcal{T}$. The scalar product $(.,.)$ is the L^2-scalar product or an equivalent one. The operator A is defined by

$$(Au, v) = \int_\Omega \nabla u \nabla v \quad \forall u, v \in V.$$

The trace operator γ associates with each function in V its restriction to Σ. The space $W = \gamma(V)$ then is the finite element space corresponding to $\mathcal{S}$ induced by V. Note, that this construction of γ and W relies on the consistency of $\mathcal{T}$ with the domain decomposition of Ω. For the scalar product $< ., . >$ we choose the scalar product of $L^2(\Sigma)$ or an equivalent one.

Let $M := \dim V$. Denote by $v_1, ..., v_M \in V$ a nodal basis of V and by $x_1, ..., x_M \in \Omega$ the corresponding nodes, i.e.

$$V = \text{span}\{v_i : 1 \leq i \leq M\} \quad \text{and} \quad v_i(x_j) = \delta_{ij} \quad \forall 1 \leq i, j \leq M.$$

Without loss of generality, we may asssume that the nodes are numbered such that

$$x_i \in \omega, \quad 1 \leq i \leq M_1, \quad x_i \in \Sigma, \quad M_1 < i \leq M.$$

We then have

$$V_1 = \ker(\gamma) = \text{span}\{v_i : 1 \leq i \leq M_1\}$$

and

$$W = \text{span}\{v_i|_\Sigma : M_1 < i \leq M\}.$$

For practical use, note that V_1 splits into N subspaces $V_{1,j}, 1 \leq j \leq N$, which are given by

$$V_{1,j} := \text{span}\{v_i : 1 \leq i \leq M_1, x_i \in \Omega_j\} \quad , 1 \leq j \leq N,$$

and which are pairwise orthogonal both with respect to the scalar products $(.,.)$ and $(A., .)$.

For the construction of E_ε we observe that problem (4.1) is a linear system of M_1 equations with M_1 unknowns. More precisely, consider an arbitrary

$$\varphi = \sum_{M_1 < i \leq M} \alpha_i v_i|_\Sigma \in W.$$

Then

$$E\varphi = \sum_{1\le i\le M} \alpha_i v_i$$

and the coefficients $\alpha_1, ..., \alpha_{M_1}$ are uniquely defined by

$$\sum_{1\le i\le M_1} \alpha_i (Av_i, v_j) = -\sum_{M_1<i\le M} \alpha_i (Av_i, v_j) \quad \forall 1 \le j \le M_1. \tag{5.1}$$

Now, we apply some stationary iterative process, such as e.g. a conjugate gradient or a multi-grid algorithm, to problem (5.1) with starting value $\alpha_1^0 = ... = \alpha_{M_1}^0 = 0$. We end this process after k iterations and obtain an approximation $\alpha_1^k, ..., \alpha_{M_1}^k$ to the solution $\alpha_1, ..., \alpha_{M_1}$ of problem (5.1). Then $E_\varepsilon\varphi$ is defined by

$$E_\varepsilon\varphi := \sum_{1\le i\le M_1} \alpha_i^k v_i + \sum_{M_1<i\le M} \alpha_i v_i. \tag{5.2}$$

Condition (4.2) is satisfied with $\varepsilon = \kappa^k$ where κ is the convergence rate of the iterative process.

Note, that the stiffness matrix of problem (5.1) is an $N \times N$ block diagonal matrix with blocks of size $\dim V_{1,j} = \sharp\{x_i \in \Omega_j\}, 1 \le j \le N$. Correspondingly, the iterative process used for the approximate solution of problem (5.1) splits into N independent subprocesses which can be performed in parallel.

Denote by $\mathcal{A}$ the stiffness matrix of A corresponding to the basis $v_1, ..., v_M$, i.e.

$$\mathcal{A}_{i,j} := (Av_i, v_j) = \int_\Omega \nabla v_i \nabla v_j \quad \forall 1 \le i, j \le M.$$

The splitting $V = V_1 \oplus \operatorname{span}\{v_i : M_1 < i \le M\}$ then induces a block decomposition of $\mathcal{A}$:

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ \mathcal{A}_{12}^T & \mathcal{A}_{22} \end{pmatrix}.$$

The Schur complement $\mathcal{A}_{22} - \mathcal{A}_{12}^T \mathcal{A}_{11}^{-1} \mathcal{A}_{12}$ is the stiffness matrix of the operator T corresponding to the basis $v_{M_1+1|\Sigma}, ..., v_{M|\Sigma}$ of W. Referring to this block decomposition, the operators E and E_ε may be represented as

$$\begin{pmatrix} -\mathcal{A}_{11}^{-1} & \mathcal{A}_{12} \\ I & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -\mathcal{R}_{11} & \mathcal{A}_{12} \\ I & 0 \end{pmatrix}.$$

Here, $\mathcal{R}_{11}$ is the approximation of $\mathcal{A}_{11}^{-1}$ which is induced by the iterative process described above. The operator T_ε is then represented by the approximate Schur complement $\mathcal{A}_{22} - \mathcal{A}_{12}^T \mathcal{R}_{11} \mathcal{A}_{12}$. This gives another interpretation of Theorem 4.2 and of the isomorphisms i and i_ε. For practical computations, however, these Schur complements and the matrix $\mathcal{R}_{11}$ will never be computed explicity.

In order to better understand how the operator $i_\varepsilon \times i^{-1}$ transforms a preconditioner for T into a preconditioner for T_ε, we consider a particular example. As in section 2 we assume that Σ does not contain any cross-points (see Fig. 1). For $1 \leq j \leq N$ we define

$$\begin{aligned} \mathcal{N}_j &:= \{i : 1 \leq i \leq M, x_i \in \overline{\Omega}_j \backslash \Gamma\}, \\ \overline{V}_{1,j} &:= \operatorname{span}\{v_{i|\Omega_j} : i \in \mathcal{N}_j\}. \end{aligned}$$

Given an arbitrary $\varphi \in W$ denote by $u_j = \sum_{i \in \mathcal{N}_j} \beta_{i,j} v_{i|\Omega_j} \in \overline{V}_{1,j}, 1 \leq j \leq N$, the unique solutions of the following N independent discrete Dirichlet-Neumann problems

$$(Au_j, v_{k|\Omega_j}) = \int_{\partial\Omega_j \backslash \Gamma} \varphi v_k \quad \forall k \in \mathcal{N}_j, 1 \leq j \leq N. \tag{5.3}$$

Let $S\varphi$ be the average of the traces of the u's on Σ. $S\varphi$ is given by

$$S\varphi = \sum_{M_1 < i \leq M} \sigma_i v_i|_\Sigma$$

with

$$\sigma_i := \frac{1}{2} \sum_{1 \leq j \leq N, i \in \mathcal{N}_j} \beta_{i,j} \quad , M_1 < i \leq M.$$

From standard trace theorems and results on elliptic regularity it follows that S is a good preconditioner for T (cf. [1]).

Now, $i_\varepsilon \times i^{-1}(S)$ is defined as follows. Apply to each of the N subproblems of (5.3) k iterations with starting value 0 of the same iterative process used for the definition of E_ε. This yields approximations $\beta_{i,j}^k, i \in \mathcal{N}_j, 1 \leq j \leq N$, for the solution $\beta_{i,j}$ of (5.3). Set

$$\sigma_i^k := \frac{1}{2} \sum_{1 \leq j \leq N, i \in \mathcal{N}_j} \beta_{i,j}^k \quad , M_1 < i \leq M.$$

Then $i_\varepsilon \times i^{-1}(S)$ is given by

$$i_\varepsilon \times i^{-1}(S)\varphi = \sum_{M_1 < i \leq M} \sigma_i^k v_i|_\Sigma.$$

5.1 Remark. The previous analysis holds for all symmetric positive definite operators on a subspace of $H^1(\Omega)$. Thus it immediately carries over to finite element discretizations of all those scalar linear elliptic equations of second order which are the Euler-Lagrange equations of a convex quadratic functional on an affine subspace of $H^1(\Omega)$. □

The Stokes equations

As our second example, we consider particular mixed finite element approximations of the Stokes equations

$$\begin{aligned} -\Delta u + \nabla p &= f \quad \text{in}\,\Omega \\ \operatorname{div} u &= 0 \quad \text{in}\,\Omega \\ u &= 0 \quad \text{on}\,\Gamma. \end{aligned}$$

Let $\mathcal{T}$ be as before. The velocity u and the pressure p are then approximated by functions in Lagrangian finite element spaces X and Y corresponding to $\mathcal{T}$. We assume that X and Y satisfy the following conditions:

(1) The discrete pressures are discontinuous and $Y \subset L^2(\Omega)$.

(2) $X \subset H(\operatorname{div}, \Omega)$ and for each $T \in \mathcal{T}$ the restrictions to T of the functions in X are contained in $H^1(T)$.

(3) X and Y satisfy the LBB-condition (cf. [4; Equ. VI.2.13]).

Assumption (2) allows the use of non-conforming finite elements. Examples of finite element spaces satisfying the above conditions are given by:

(a) the $P2/P0$-element consisting of continuous, piecewise quadratic velocities and piecewise constant pressures,

(b) the $Pk/P(k-1)$-element, $k \geq 4$, consisting of velocities, which are continuous and piecewise polynomials of degree k, and of pressures, which are discontinuous and piecewise polynomials of degree $k-1$,

(c) the Crouzeix-Raviart element consisting of piecewise linear velocities, which are continuous at the midpoints of the triangles' edges, and of piecewise constant pressures.

Note, that the Taylor-Hood element and the mini-element, which both use continuous pressure-approximations, do not satisfy condition (1) above.

As before, (.,.) denotes the L^2-scalar product or an equivalent one. V is the space of all discrete-solenoidal functions:

$$V := \{u \in X : \int_\Omega p \operatorname{div} u = 0 \quad \forall p \in Y\}.$$

Note that in the examples (b) and (c) the functions in V are exactly solenoidal. The operator A is defined by

$$(Au, v) := \sum_{T \in \mathcal{T}} \int_T \nabla u \nabla v.$$

This definition makes sense thanks to condition (2) above. The trace operator γ, the space W, and the operators E, E_ε, T, and T_ε are defined as before.

Problems (5.1) and (5.3) are now discrete Stokes problems. For their pratical solution it is important that they still split into N independent discrete Stokes problems on the subdomains $\Omega_j, 1 \leq j \leq N$. This is so since condition (1) above implies that

$$p \in Y \Rightarrow p\chi_{\Omega_j} \in Y \quad \forall 1 \leq j \leq N$$

and therefore

$$u \in V_1 \Rightarrow u\chi_{\Omega_j} \in V_1 \quad \forall 1 \leq j \leq N. \tag{5.4}$$

Here, χ_{Ω_j} denotes the characteristic function of Ω_j. Thanks to (5.4) V_1 splits into N subspaces

$$V_{1,j} := \{u\chi_{\Omega_j} : u \in V_1\} \quad , 1 \leq j \leq N,$$

which are pairwise orthogonal both with respect to the scalar products (.,.) and (A.,.). For pratical use note that - thanks to condition (1) above - $V_{1,j}$ can equivalently be characterized by

$$\begin{aligned} V_{1,j} &= \{u \in X : \int_\Omega p \operatorname{div} u = 0 \ \forall p \in Y \ , \ u = 0 \text{ on } \Omega \backslash \Omega_j\} \\ &= \{u|_{\Omega_j} : u \in X, u = 0 \text{ on } \partial\Omega_j, \int_{\Omega_j} p \operatorname{div} u = 0 \quad \forall p \in Y\}. \end{aligned}$$

The following example shows that this characterization and the property (5.4) are violated when using continuous pressure approximations. Consider a rectangle Ω with sides of length 2 which is cut into two unit squares Ω_1 and Ω_2 as depicted in Fig. 3. As discretization choose the Taylor-Hood element which consists of continuous piecewise quadratic velocities and continuous piecewise linear pressures. The velocities are uniquely determined by their values at the vertices and the midpoints of the edges. Let u be the velocity which corresponds to a unit vortex around the midpoint of Σ. In the midpoints of edges, which are marked in Fig. 3, it is a unit vector tangent to the corresponding edge. In all other midpoints of edges and in all vertices u vanishes. A straightforward calculation yields that $u \in V$ and $u = 0$ on Σ but that $u\chi_{\Omega_j} \notin V_{1,j}, j = 1, 2$.

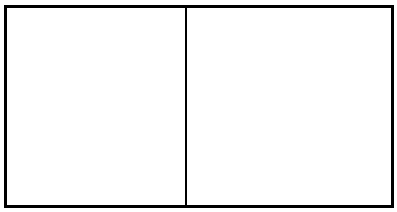

Fig. 3: Taylor-Hood discretization of the Stokes equations; the velocity equals a unit tangent vector at the marked midpoints of edges and vanishes at all other midpoints of edges and at all vertices

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