RUB

A Posteriori Error Analysis of Space-Time Finite Element Discretizations of the Time-Dependent Stokes Equations

R. Verfürth

Fakultät für Mathematik
Ruhr-Universität Bochum

www.ruhr-uni-bochum.de/num1

Milan / February 14th, 2011

RUB

Outline

Variational Problem

Discretization

A Posteriori Error Analysis

RUB

Time-Dependent Stokes Equations

$$\begin{aligned} \partial_t \mathbf{u} - \nu \Delta \mathbf{u} + \nabla p &= \mathbf{f} && \text{in } \Omega \times (0,T) \\ \operatorname{div} \mathbf{u} &= 0 && \text{in } \Omega \times (0,T) \\ \mathbf{u} &= 0 && \text{on } \Gamma \times (0,T) \\ \mathbf{u}(\cdot,0) &= \mathbf{u}_0 && \text{in } \Omega \end{aligned}$$

RUB

Variational Formulation

Find $\mathbf{u} \in L^2(0,T;H^1_0(\Omega)^d) \cap L^\infty(0,T;L^2(\Omega)^d)$ with $\partial_t \mathbf{u} \in L^2(0,T;H^{-1}(\Omega)^d)$ and $p \in L^2(0,T;L^2_0(\Omega))$ such that for almost all $t \in (0,T)$ and all $\mathbf{v} \in H^1_0(\Omega)^d$, $q \in L^2_0(\Omega)$

$$\mathbf{u}(\cdot,0) = \mathbf{u}_0 \quad \text{in } H^{-1}(\Omega)^d$$

and

$$\begin{aligned} \int_\Omega \partial_t \mathbf{u} \cdot \mathbf{v} + \nu \int_\Omega \nabla \mathbf{u} : \nabla \mathbf{v} - \int_\Omega p \operatorname{div} \mathbf{v} &= \int_\Omega \mathbf{f} \cdot \mathbf{v}, \\ \int_\Omega q \operatorname{div} \mathbf{u} &= 0. \end{aligned}$$

Stability

Inserting $\mathbf{u}$ as a test-function in the variational formulation and taking into account that $\operatorname{div}\mathbf{u} = 0$ yields

$$\left\{\|\partial_t\mathbf{u} + \nabla p\|^2_{L^2(H^{-1})} + \|\mathbf{u}\|^2_{L^\infty(L^2)} + \nu\|\mathbf{u}\|^2_{L^2(H^1)}\right\}^{\frac{1}{2}} \le \left\{(4 + \frac{2}{\nu})\|\mathbf{f}\|^2_{L^2(H^{-1})} + (4\nu + 2)\|\mathbf{u}_0\|^2_{L^2}\right\}^{\frac{1}{2}}.$$

Partitions and Spaces

- $\mathcal{I} = \{(t_{n-1}, t_n] : 1 \le n \le N_\mathcal{I}\}$ partition of $[0, T]$.
- $\tau_n = t_n - t_{n-1}$.
- $\mathcal{T}_n$, $0 \le n \le N_\mathcal{I}$, affine equivalent, admissible, shape regular partitions of Ω.
- Modified transition condition: There are two partitions $\mathcal{T}'_n$ and $\mathcal{T}''_n$ such that $\mathcal{T}_n$ and $\mathcal{T}_{n-1}$ are refinements of $\mathcal{T}'_n$ and sucht that $\mathcal{T}''_n$ is a refinement of $\mathcal{T}_n$ and $\mathcal{T}_{n-1}$ and such that $h_{K'} \lesssim h_K \lesssim h_{K''}$ holds for all $K' \subset K \subset K''$ uniformly for all n.
- $V_n \subset H^1_0(\Omega)$, $P_n \subset L^2_0(\Omega)$ finite element spaces corresponding to $\mathcal{T}_n$.

Discrete Problem

Find $\mathbf{u}^n_{\mathcal{T}_n} \in V_n$, $p^n_{\mathcal{T}_n} \in P_n$ such that $\mathbf{u}^0_{\mathcal{T}_0} = \pi_0\mathbf{u}_0$ and for all $\mathbf{v}_{\mathcal{T}_n} \in V_n$, $q_{\mathcal{T}_n} \in P_n$ with $\mathbf{u}^{n\theta} = \theta\mathbf{u}^n_{\mathcal{T}_n} + (1-\theta)\mathbf{u}^{n-1}_{\mathcal{T}_{n-1}}$

$$\begin{aligned}
&\int_\Omega \frac{1}{\tau_n}(\mathbf{u}^n_{\mathcal{T}_n} - \mathbf{u}^{n-1}_{\mathcal{T}_{n-1}}) \cdot \mathbf{v}_{\mathcal{T}_n} + \nu\int_\Omega \nabla\mathbf{u}^{n\theta} : \nabla\mathbf{v}_{\mathcal{T}_n} \\
&- \int_\Omega p^n_{\mathcal{T}_n} \operatorname{div}\mathbf{v}_{\mathcal{T}_n} + \int_\Omega q_{\mathcal{T}_n} \operatorname{div}\mathbf{u}^n_{\mathcal{T}_n} \\
&+ \sum_{K\in\mathcal{T}_n} \vartheta_K h_K^2 \int_K \left[\frac{\mathbf{u}^n_{\mathcal{T}_n} - \mathbf{u}^{n-1}_{\mathcal{T}_{n-1}}}{\tau_n} - \nu\Delta\mathbf{u}^{n\theta} + \nabla p^n_{\mathcal{T}_n}\right] \cdot \nabla q_{\mathcal{T}_n} \\
&+ \sum_{E\in\mathcal{E}_n} \vartheta_E h_E \int_E \mathbb{J}_E(p^n_{\mathcal{T}_n})\mathbb{J}_E(q_{\mathcal{T}_n}) + \sum_{K\in\mathcal{T}_n} \widetilde{\vartheta}_K \int_K \operatorname{div}\mathbf{u}^n_{\mathcal{T}_n} \operatorname{div}\mathbf{v}_{\mathcal{T}_n} \\
= &\int_\Omega \mathbf{f}^{n\theta} \cdot \mathbf{u}_{\mathcal{T}_n} + \sum_{K\in\mathcal{T}_n} \vartheta_K h_K^2 \int_K \mathbf{f}^{n\theta} \cdot \nabla q_{\mathcal{T}_n}
\end{aligned}$$

Examples of Spaces V_n and P_n

- Without stabilization:
  - Mini element
  - Hood-Taylor element
  - Modified Hood-Taylor element
  - Higher order Hood-Taylor elements
  - Bernardi-Raugel element
- With stabilization:
  - Equal order interpolation
  - Continuous velocities of order k and discontinuous pressures of order $k-1$

Basic Steps

- Error and residual are equivalent.
- The residual splits into a spatial and a temporal residual.
- The norm of the sum of these is equivalent to the sum of their norms.
- Derive an error indicator for the spatial residual.
- Derive an error indicator for the temporal residual.
- The first step requires properties of the Stokes projection.

Stokes Projection

- Stokes projection $\Pi\mathbf{v} : H_0^1(\Omega)^d \to V^\perp$:
  $\int_\Omega \nabla\Pi\mathbf{v} : \nabla\mathbf{w} - \int_\Omega q \operatorname{div}\mathbf{w} = 0, \int_\Omega r \operatorname{div}\Pi\mathbf{v} = \int_\Omega r \operatorname{div}\mathbf{v}$
- For all $\mathbf{v} \in H_0^1(\Omega)^d$:
  $\|\nabla\Pi\mathbf{v}\| \le \frac{1}{\beta}\|\operatorname{div}\mathbf{v}\|$
  with β the analytical inf-sup constant.
- If $\int_\Omega q_\mathcal{T} \operatorname{div}\mathbf{v} = 0$ for all piecewise constant or all continuous piecewise linear $q_\mathcal{T}$:
  $\|\nabla\Pi\mathbf{v}\| \le c_\Pi \Big\{ \sum_{K\in\mathcal{T}} h_K^{2\alpha_K} \|\operatorname{div}\mathbf{v}\|_K^2 \Big\}^{\frac{1}{2}}$
  with $\alpha_K = 1$ if K does not contain a re-enetrant corner and $\alpha_K = \frac{1}{2}$ otherwise.

Sketch of Proof

- First part:
  - Insert $\Pi\mathbf{v}$ as test function in the defining equations and use the stability of the divergence operator.
- Second part:
  - Insert $\Pi\mathbf{v}$ as test function in the dual Stokes problem
    $\int_\Omega \nabla\mathbf{z} : \nabla\mathbf{w} - \int_\Omega s \operatorname{div}\mathbf{w} = \int_\Omega \Pi\mathbf{v}\cdot\mathbf{w}, \int_\Omega r \operatorname{div}\mathbf{z} = 0.$
  - Use approximation properties of the L^2-projection onto piecewise constant or continuous piecewise linear functions and regularity results for the dual Stokes problem.

Errors and Residuals

- $\mathbf{u}_\mathcal{I}$: continuous piecewise linear w.r.t. time equals $\mathbf{u}_{\mathcal{T}_n}^n$ at time t_n
- Velocity error: $\mathbf{e} = \mathbf{u} - \mathbf{u}_\mathcal{I}$
- $p_\mathcal{I}$: piecewise constant w.r.t. time equals $p_{\mathcal{T}_n}^n$ on $(t_{n-1}, t_n]$
- Pressure error: $\varepsilon = p - p_\mathcal{I}$
- Residual of momentum equation:
  $\langle R_\mathrm{m}, \mathbf{v}\rangle = \int_\Omega \big(\mathbf{f}\cdot\mathbf{v} - \partial_t\mathbf{u}_\mathcal{I}\cdot\mathbf{v} - \nu\nabla\mathbf{u}_\mathcal{I} : \nabla\mathbf{v} + p_\mathcal{I}\operatorname{div}\mathbf{v}\big)$
- Residual of continuity equation:
  $\langle R_\mathrm{c}, q\rangle = -\int_\Omega q \operatorname{div}\mathbf{u}_\mathcal{I}$

Equivalence of Error and Residual

- Lower bound:

$$\|R_{\mathrm{m}}\|_{L^2(H^{-1})} + \|R_{\mathrm{c}}\|_{L^2(L^2)}$$
$$\lesssim \{\|\partial_t \mathbf{e} + \nabla\varepsilon\|^2_{L^2(H^{-1})} + \|\mathbf{e}\|^2_{L^\infty(L^2)} + \frac{1}{\nu}\nu\|\mathbf{e}\|^2_{L^2(H^1)}\}^{\frac{1}{2}}$$

- Upper bound:

$$\{\|\partial_t \mathbf{e} + \nabla\varepsilon\|^2_{L^2(H^{-1})} + \|\mathbf{e}\|^2_{L^\infty(L^2)} + \nu\|\mathbf{e}\|^2_{L^2(H^1)}\}^{\frac{1}{2}}$$
$$\lesssim \{\frac{1}{\nu}\|R_{\mathrm{m}}\|^2_{L^2(H^{-1})} + \|R_{\mathrm{c}}\|^2_{L^2(L^2)} + \|\mathbf{e}_0\|^2$$
$$+ \max_{0\le n\le N_{\mathcal{I}}} \sum_{K\in\mathcal{T}_n} h_K^{2\alpha_K}\|\operatorname{div}\mathbf{u}^n_{\mathcal{T}_n}\|^2_K$$
$$+ \big(\sum_{n=1}^{N_{\mathcal{I}}}\big[\sum_{K\in\mathcal{T}_n} h_K^{2\alpha_K}\|\operatorname{div}(\mathbf{u}^n_{\mathcal{T}_n} - \mathbf{u}^{n-1}_{\mathcal{T}_{n-1}})\|^2_K\big]^{\frac{1}{2}}\big)^2\}^{\frac{1}{2}}$$

Sketch of Proof

- Lower bound:
  - Follows from the definition of the errors and residuals and the Cauchy-Schwarz inequality.
- Upper bound:
  - Inserting $\mathbf{e}-\Pi\mathbf{u}_{\mathcal{I}}$ in the definition of R_{m} yields $\frac{d}{dt}\|\mathbf{e}\|^2 + \nu\|\nabla\mathbf{e}\|^2 \lesssim \|R_{\mathrm{m}}\|^2_{H^{-1}} + \|R_{\mathrm{c}}\|^2 - 2\langle\partial_t(\mathbf{u}-\mathbf{u}_{\mathcal{I}}), \Pi\mathbf{u}_{\mathcal{I}}\rangle$.
  - The term involving $\Pi\mathbf{u}_{\mathcal{I}}$ is controlled using the properties of the Stokes projection.
  - Integration w.r.t. time yields the upper bound.

Decomposition of Residuals

- Spatial residuals:
  - $\langle R_{\mathrm{m},h}, \mathbf{v}\rangle = \int_\Omega \big(\mathbf{f}^{n\theta}\cdot\mathbf{v} - \partial_t\mathbf{u}_{\mathcal{I}}\cdot\mathbf{v} - \nu\nabla\mathbf{u}^{n\theta} : \nabla\mathbf{v} + p^n_{\mathcal{T}_n}\operatorname{div}\mathbf{v}\big)$
  - $\langle R_{\mathrm{c},h}(\mathbf{u}_{\mathcal{I}}, p_{\mathcal{I}}), q\rangle = -\int_\Omega q\operatorname{div}\mathbf{u}^n_{\mathcal{T}_n}$
- Temporal residuals:
  - $\langle R_{\mathrm{m},\tau}, \mathbf{v}\rangle = \nu\int_\Omega \nabla\big[\mathbf{u}^{n\theta} - \mathbf{u}_{\mathcal{I}}\big] : \nabla\mathbf{v}$
  - $\langle R_{\mathrm{c},\tau}, q\rangle = \int_\Omega q\operatorname{div}\big[\mathbf{u}^n_{\mathcal{T}_n} - \mathbf{u}_{\mathcal{I}}\big]$
- Decomposition:
  - $R_{\mathrm{m}} = R_{\mathrm{m},h} + R_{\mathrm{m},\tau}$, $R_{\mathrm{c}} = R_{\mathrm{c},h} + R_{\mathrm{c},\tau}$
- Sharpness of the triangle inequality: The norms of R_{m} and R_{c} are equivalent to the sums of the norms of $R_{\mathrm{m},\tau}$, $R_{\mathrm{m},h}$ and $R_{\mathrm{c},\tau}$, $R_{\mathrm{c},h}$, resp.

Sharpness of the Triangle Inequality

For every Banach space Y, elements $\varphi, \psi \in Y^*$, interval (a,b) and parameter $\theta \in [\frac{1}{2}, 1]$ there holds

$$\sqrt{\frac{5}{14}}\big(1 - \frac{\sqrt{3}}{2}\big)\{\|\varphi\|^2_{L^2(Y^*)} + \|(\theta - \frac{t-a}{b-a})\psi\|^2_{L^2(Y^*)}\}^{\frac{1}{2}}$$
$$\le \|\varphi + (\theta - \frac{t-a}{b-a})\psi\|_{L^2(Y^*)}$$
$$\le \|\varphi\|_{L^2(Y^*)} + \|(\theta - \frac{t-a}{b-a})\psi\|_{L^2(Y^*)}$$

Sketch of Proof

- Only the lower bound has to be proven.
- A scaling argument shows that w.l.o.g. $a = 0$, $b = 1$.
- Choose $v, w \in Y$ such that $\langle \varphi, v\rangle = \|\varphi\|_*^2$, $\|v\|_Y = \|\varphi\|_*$, $\langle \psi, w\rangle = \|\psi\|_*^2$, $\|w\|_Y = \|\psi\|_*$.
- Hölder's inequality yields $\|3t^2 v + (\theta - t)w\|_{L^2(Y)} \le \sqrt{\frac{14}{5}}\big\{\|\varphi\|_{L^2(Y^*)}^2 + \|(\theta - t)\psi\|_{L^2(Y^*)}^2\big\}^{\frac{1}{2}}$.
- Applying the inequality $-ab \ge -\frac{a^2}{2} - \frac{b^2}{2}$ twice gives
$$\int_0^1 \langle \varphi + (\theta - t)\psi, 3t^2 v + (\theta - t)w\rangle dt \ge$$
$$(1 - \frac{\sqrt{3}}{2})\big\{\|\varphi\|_{L^2(Y^*)}^2 + \|(\theta - t)\psi\|_{L^2(Y^*)}^2\big\}^{\frac{1}{2}}.$$

Estimation of the Spatial Residual

- The spatial residual is the residual of a standard discretization of a stationary Stokes problem.
- Standard techniques for stationary problems yield
  - $\|R_{\mathrm{m},h}\|_{H^{-1}} + \|R_{\mathrm{c},h}\| \le c^*\{\eta_h^n + \Theta_h^n\}$
  - $\eta_h^n \le c_*\{\|R_{\mathrm{m},h}\|_{H^{-1}} + \|R_{\mathrm{c},h}\| + \Theta_h^n\}$

  with
  - $\eta_h^n = \Big\{\sum_K h_K^2 \|\mathbf{f}_{\mathcal{T}_n}^{n\theta} - \partial_t \mathbf{u}_{\mathcal{I}} + \nu\Delta \mathbf{u}^{n\theta} - \nabla p_{\mathcal{T}_n}^n\|_K^2 + \sum_E h_E \|\mathbb{J}_E(\mathbf{n}_E \cdot (\nu\nabla \mathbf{u}^{n\theta} - p_{\mathcal{T}_n}^n I))\|_E^2 + \sum_K \|\operatorname{div} \mathbf{u}_{\mathcal{T}_n}^n\|_K^2\Big\}^{\frac{1}{2}}$
  - $\Theta_h^n = \Big\{\sum_K h_K^2 \|\mathbf{f} - \mathbf{f}_{\mathcal{T}_n}^{n\theta}\|_K^2\Big\}^{\frac{1}{2}}$.

Estimation of the Temporal Residual

- R_{m} is piecewise linear w.r.t. time.
- R_{c} is piecewise constant w.r.t. time.
- Exact integration w.r.t. time yields
  - $\sqrt{\frac{\tau_n}{12}}\|\nabla(\mathbf{u}_{\mathcal{T}_n}^n - \mathbf{u}_{\mathcal{T}_{n-1}}^{n-1})\| \le \|R_{\mathrm{m},\tau}\|_{L^2(H^{-1})} \le \sqrt{\frac{\tau_n}{3}}\|\nabla(\mathbf{u}_{\mathcal{T}_n}^n - \mathbf{u}_{\mathcal{T}_{n-1}}^{n-1})\|$
  - $\|R_{\mathrm{c},\tau}\|_{L^2(L^2)} = \sqrt{\frac{\tau_n}{3}}\|\operatorname{div}(\mathbf{u}_{\mathcal{T}_n}^n - \mathbf{u}_{\mathcal{T}_{n-1}}^{n-1})\|$.
- Set
  - $\eta_\tau^n = \Big\{\|\nabla(\mathbf{u}_{\mathcal{T}_n}^n - \mathbf{u}_{\mathcal{T}_{n-1}}^{n-1})\|^2 + \|\operatorname{div}(\mathbf{u}_{\mathcal{T}_n}^n - \mathbf{u}_{\mathcal{T}_{n-1}}^{n-1})\|^2\Big\}^{\frac{1}{2}}$.

A Posteriori Error Estimates

- Upper bound:
$$\Big\{\|\partial_t \mathbf{e} + \nabla\varepsilon\|_{L^2(H^{-1})}^2 + \|\mathbf{e}\|_{L^\infty(L^2)}^2 + \nu\|\mathbf{e}\|_{L^2(H^1)}^2\Big\}^{\frac{1}{2}}$$
$$\lesssim \Big\{\sum_n \tau_n\big[(\eta_\tau^n)^2 + (\eta_h^n)^2 + (\Theta_h^n)^2\big] + \|\mathbf{u}_0 - \mathbf{u}_{\mathcal{T}_0}^0\|^2 + \max_n \sum_K h_K^{2\alpha_K}\|\operatorname{div} \mathbf{u}_{\mathcal{T}_n}^n\|_K^2 + \Big(\sum_n \Big[\sum_K h_K^{2\alpha_K}\|\operatorname{div}(\mathbf{u}_{\mathcal{T}_n}^n - \mathbf{u}_{\mathcal{T}_{n-1}}^{n-1})\|_K^2\Big]^{\frac{1}{2}}\Big)^2\Big\}^{\frac{1}{2}}$$
- Lower bound:
$$\Big\{\sum_n \tau_n\big[(\eta_\tau^n)^2 + (\eta_h^n)^2\big]\Big\}^{\frac{1}{2}}$$
$$\lesssim \Big\{\|\partial_t \mathbf{e} + \nabla\varepsilon\|_{L^2(H^{-1})}^2 + \|\mathbf{e}\|_{L^\infty(L^2)}^2 + \nu\|\mathbf{e}\|_{L^2(H^1)}^2 + \|\mathbf{u}_0 - \mathbf{u}_{\mathcal{T}_0}^0\|^2 + \sum_n \tau_n(\Theta_h^n)^2\Big\}^{\frac{1}{2}}$$

Comments

- The terms $\sqrt{\tau_n}\eta_h^n$ control the spatial error and can be used to adapt the spatial meshes.
- The terms $\sqrt{\tau_n}\eta_\tau^n$ control the temporal error and can be used to adapt the time-steps.
- If $h_K^{2\alpha_K} \le \tau_n$ holds for all K and n, $\max_n \sum_K h_K^{2\alpha_K} \|\text{div}\, \mathbf{u}_{\mathcal{T}_n}^n\|_K^2$ can be absorbed by $\sum_n \tau_n \big(\eta_h^n\big)^2$. This also holds in the presence of re-entrant corners.
- If $h_K^{2\alpha_K} \le \tau_n^2$ holds for all K and n, $\Big(\sum_n \Big[\sum_K h_K^{2\alpha_K} \|\text{div}(\mathbf{u}_{\mathcal{T}_n}^n - \mathbf{u}_{\mathcal{T}_{n-1}}^{n-1})\|_K^2\Big]^{\frac{1}{2}}\Big)^2$ can be absorbed by $T \sum_n \tau_n \big(\eta_\tau^n\big)^2$. This does not hold in the presence of re-entrant corners.

References

- (with Ch. Bernardi) A posteriori error analysis of the fully discretized time-dependent Stokes equations,
M2AN **38** (2004), no. 3, 437 – 455
- A posteriori error estimates for finite element discretizations of the heat equation,
Calcolo **40** (2003), no. 3, 195 – 212.
- Robust a posteriori error estimates for nonstationary convection-diffusion equations,
SIAM J. Numer. Anal. **43** (2005), no. 4, 1783 – 1802
- A posteriori error analysis of space-time finite element discretizations of the time-dependent Stokes equations,
Calcolo **47** (2010), 149 – 167