Adaptive Finite Element Methods for Non-Stationary Convection-Diffusion Problems

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Goal

- Present space-time adaptive finite element methods for non-stationary convection-diffusion equations based on stable discretizations and a posteriori error estimates.
- A posteriori error estimates should yield upper and lower bounds for the energy norm of the error that are uniform with respect to all possible relative sizes of convection to diffusion.
- Use a common framework for various stabilization methods.

Outline

Differential Equation

$$\begin{aligned} \partial_t u - \varepsilon\Delta u + \mathbf{a}\cdot\nabla u + \beta u &= f && \text{in } \Omega\times(0,T] \\ u &= 0 && \text{on } \Gamma\times(0,T] \\ u &= u_0 && \text{in } \Omega \end{aligned}$$

- $0 < \varepsilon \le 1$, $\beta \ge 0$, $\mathbf{a} \in \mathbb{R}^d$, $|\mathbf{a}| = 1$, f piecewise polynomial.
- Results hold for general f, variable coefficients, and mixed boundary conditions. Then ε is a lower bound for the smallest eigenvalue of the diffusion and β is a lower bound for $b - \frac{1}{2}\operatorname{div}\mathbf{a}$ with b denoting the reaction.
- General right-hand sides f and variable coefficients give rise to additional oscillation terms.
- All estimates should be uniform w.r.t. ε.

Variational Problem

Find $u \in L^2(0,T;H_0^1(\Omega))$ with $\partial_t u \in L^2(0,T;H^{-1}(\Omega))$ such that $u = u_0$ in L^2 and for all $t \in (0,T)$ and all $v \in H_0^1(\Omega)$

$$\langle \partial_t u\,,\, v\rangle + \underbrace{\int_\Omega \{\varepsilon \nabla u \cdot \nabla v + \mathbf{a}\cdot\nabla u v + \beta u v\}}_{=B(u,v)} = \underbrace{\int_\Omega f v}_{=\langle \ell\,,\, v\rangle}$$

Norms

- Energy norm $|||v||| = \left\{\varepsilon\,\|\nabla v\|^2 + \beta\,\|v\|^2\right\}^{\frac{1}{2}}$
- Dual norm $\quad |||\varphi|||_* = \sup\limits_{v\in H_0^1(\Omega)\setminus\{0\}} \dfrac{\langle\varphi, v\rangle}{|||v|||}$
- Error norm

$$\|u\|_{X(a,b)} = \left\{ \operatorname*{ess.\,sup}_{t\in(a,b)} \|u(\cdot,t)\|^2 + \int_a^b |||u(\cdot,t)|||^2\,dt \right.$$
$$\left. + \int_a^b |||(\partial_t u + \mathbf{a}\cdot\nabla u)(\cdot,t)|||_*^2\,dt \right\}^{\frac{1}{2}}$$

Meshes and Spaces

- $\mathcal{I} = \{(t_{n-1}, t_n] : 1 \le n \le N_\mathcal{I}\}$ partition of $[0,T]$.
- $\tau_n = t_n - t_{n-1}$.
- $\mathcal{T}_n$, $0 \le n \le N_\mathcal{I}$, affine equivalent, admissible, shape regular partitions of Ω.
- Transition condition: There is a common refinement $\widetilde{\mathcal{T}}_n$ of $\mathcal{T}_n$ and $\mathcal{T}_{n-1}$ such that $h_K \le c h_{K'}$ for all $K \in \mathcal{T}_n$ and all $K' \in \widetilde{\mathcal{T}}_n$ with $K' \subset K$.
- $X_n \subset H_0^1(\Omega)$ finite element space corresponding to $\mathcal{T}_n$.

Discrete Problem

Find $u_{\mathcal{T}_n}^n \in X_n$, $0 \le n \le N_\mathcal{I}$, such that $u_{\mathcal{T}_0}^0 = \pi_0 u_0$ and, for $n = 1, \dots, N_\mathcal{I}$ and all $v_{\mathcal{T}_n} \in X_n$ with $U^{n\theta} = \theta \nabla u_{\mathcal{T}_n}^n + (1-\theta)\nabla u_{\mathcal{T}_{n-1}}^{n-1}$

$$\left(\frac{u_{\mathcal{T}_n}^n - u_{\mathcal{T}_{n-1}}^{n-1}}{\tau_n}\,,\, v_{\mathcal{T}_n}\right) + B(U^{n\theta}, v_{\mathcal{T}_n}) + S_n(U^{n\theta}, v_{\mathcal{T}_n}) = \langle \ell\,,\, v_{\mathcal{T}_n}\rangle$$

- The stabilization term S_n is supposed to be linear in its second argument and affine in its first argument, it may depend on $\mathcal{T}_n$ and on f.
- Solution $u_\mathcal{I}$ is continuous piece-wise affine and equals $u_{\mathcal{T}_n}^n$ at t_n.

Stabilizations

- Streamline diffusion method
  $S_n(u,v) = \sum_K \vartheta_K \int_K \{\partial_t u - \varepsilon\Delta u + \mathbf{a}\cdot\nabla u + \beta u - f\}\mathbf{a}\cdot\nabla v$
  with $\vartheta_K\,|\mathbf{a}| \le c h_K$
- Continuous interior penalty method
  $S_n(u,v) = \sum_E \vartheta_E \int_E \mathbb{J}_E(\mathbf{a}\cdot\nabla u)\mathbb{J}_E(\mathbf{a}\cdot\nabla v)$
  with $\vartheta_E \le c h_E^2$
- Local projection scheme
  $S_n(u,v) = \sum_M \vartheta_M \int_M \kappa_{\mathcal{M}}(\mathbf{a}\cdot\nabla u)\,\kappa_{\mathcal{M}}(\mathbf{a}\cdot\nabla v)$
  with $\vartheta_M\,|\mathbf{a}| \le c h_M$ and $I-\kappa_{\mathcal{M}}$ projection onto $S^{\ell,-1}(\mathcal{M})$
- Subgrid scale approach
  $S_n(u,v) = \sum_K \vartheta_K \int_K \mathbf{a}\cdot\nabla\Pi_n(u)\;\mathbf{a}\cdot\nabla\Pi_n(v)$
  with $\vartheta_K\,|\mathbf{a}| \le c h_K$ and Π_n projection onto $Y_n \subset X_n$

Basic Steps

- Error and residual are equivalent.
- The residual splits into a spatial and a temporal residual.
- The norm of the sum of these is equivalent to the sum of their norms.
- Derive a reliable, efficient and robust error indicator for the temporal residual.
- Derive a reliable, efficient and robust error indicator for the spatial residual.
- All stabilizations yield the same spatial error indicator.

Equivalence of Error and Residual

- $u_{\mathcal{I}}$ is continuous piece-wise affine and equals $u^n_{\mathcal{T}_n}$ at t_n.
- Residual:

  $\langle R(u_{\mathcal{I}})\,,\,v\rangle = \langle \ell\,,\,v\rangle - \langle \partial_t u_{\mathcal{I}}\,,\,v\rangle - B(u_{\mathcal{I}},v)$
- Lower error-bound:

  $\|R(u_{\mathcal{I}})\|_{L^2(t_{n-1},t_n;H^{-1})} \le \sqrt{2}\,\|u-u_{\mathcal{I}}\|_{X(t_{n-1},t_n)}$
- Upper error-bound:

  $\|u-u_{\mathcal{I}}\|_{X(0,t_n)} \le \left\{4\,\|u_0-\pi_0 u_0\|^2 + 6\,\|R(u_{\mathcal{I}})\|^2_{L^2(0,t_n;H^{-1})}\right\}^{\frac{1}{2}}$

Proof of the Equivalence

- Relation of residual and error:

  $\langle R(u_{\mathcal{I}})\,,\,v\rangle = \langle \partial_t e\,,\,v\rangle + B(e,v)$
- Lower error-bound: Definition of primal and dual norm plus Cauchy-Schwarz inequality.
- Upper error-bound: Parabolic energy estimate with $v=e$ as test-function.

Decomposition of the Residual

- Recall $U^{n\theta} = \theta \nabla u^n_{\mathcal{T}_n} + (1-\theta)\nabla u^{n-1}_{\mathcal{T}_{n-1}}$
- Temporal residual:

  $\langle R_\tau(u_\mathcal{I}), v\rangle = B(U^{n\theta} - u_\mathcal{I}, v)$
- Spatial residual:

  $\langle R_h(u_\mathcal{I}), v\rangle = \langle \ell, v\rangle - \langle \partial_t u_\mathcal{I}, v\rangle - B(U^{n\theta}, v)$
- Splitting: $R(u_\mathcal{I}) = R_\tau(u_\mathcal{I}) + R_h(u_\mathcal{I})$
- Estimate for $L^2(t_{n-1}, t_n; H^{-1})$-norms:

$$\frac{1}{13}\left\{\|R_\tau(u_\mathcal{I})\|^2 + \|R_h(u_\mathcal{I})\|^2\right\}^{\frac{1}{2}} \le \|R_\tau(u_\mathcal{I}) + R_h(u_\mathcal{I})\| \le \|R_\tau(u_\mathcal{I})\| + \|R_h(u_\mathcal{I})\|$$

Motivation of the Lower Bound

- Strengthened Cauchy-Schwarz inequality for $v = c$ and $w = \frac{b-t}{b-a}$:

$$\int_a^b vw = \frac{1}{2}c(b-a) = \frac{\sqrt{3}}{2}\|v\|_{(a,b)}\|w\|_{(a,b)}$$
- Hence:

$$\|v+w\|^2_{(a,b)} \ge \left(1 - \frac{\sqrt{3}}{2}\right)\left\{\|v\|^2_{(a,b)} + \|w\|^2_{(a,b)}\right\}$$

Proof of the Lower Bound

- $R_h(u_\mathcal{I})$ is piece-wise constant.
- $R_\tau(u_\mathcal{I})$ is piece-wise affine: $R_\tau(u_\mathcal{I}) = \left(\theta - \frac{t-t_{n-1}}{\tau_n}\right)\rho^n$ with

  $\langle \rho^n, v\rangle = B(u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}}, v)$.
- Choose $v, w \in H^1_0(\Omega)$ such that

  $|||v||| = |||R_h(u_\mathcal{I})|||_*, \quad \langle R_h(u_\mathcal{I}), v\rangle = |||R_h(u_\mathcal{I})|||^2_*,$

  $|||w||| = |||\rho^n|||_*, \quad \langle \rho^n, w\rangle = |||\rho^n|||^2_*.$
- Insert $3\left(\frac{t-t_{n-1}}{\tau_n}\right)^2 v + \frac{t_n - t}{\tau_n} w$ as test-function in representation of $R(u_\mathcal{I})$.

Estimation of the Temporal Residual

- $R_\tau(u_\mathcal{I}) = \left(\theta - \frac{t-t_{n-1}}{\tau_n}\right)\rho^n$ with

  $\langle \rho^n, v\rangle = B(u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}}, v)$.
- Upper bound:

$$|||\rho^n|||_* \le \left|\left|\left| u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}} \right|\right|\right| + \left|\left|\left| \mathbf{a}\cdot\nabla(u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}}) \right|\right|\right|_*$$
- Follows from definition of ρ^n and $|||\cdot|||_*$.
- Lower bound:

$$\frac{1}{3}\left\{\left|\left|\left| u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}} \right|\right|\right| + \left|\left|\left| \mathbf{a}\cdot\nabla(u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}}) \right|\right|\right|_*\right\} \le |||\rho^n|||_*$$

Proof of the Lower Bound

- Set $w^n = u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}}$ and choose $v \in H^1_0(\Omega)$ with $|||v||| = |||\mathbf{a}\cdot\nabla w^n|||_*$ and $(\mathbf{a}\cdot\nabla w^n\,,\,v) = |||\mathbf{a}\cdot\nabla w^n|||_*^2$
- Insert $\frac{1}{2}w^n + \frac{1}{2}v$ in the definition of ρ^n:

$$\begin{aligned}
&\left\langle \rho^n\,,\,\frac{1}{2}w^n + \frac{1}{2}v \right\rangle \\
&= \underbrace{\frac{1}{2}(\varepsilon\nabla w^n\,,\,\nabla w^n) + \frac{1}{2}(\beta w^n\,,\,w^n)}_{=\frac{1}{2}|||w^n|||^2} + \underbrace{\frac{1}{2}(\mathbf{a}\cdot\nabla w^n, w^n)}_{=0} \\
&\quad + \underbrace{\frac{1}{2}(\varepsilon\nabla w^n\,,\,\nabla v) + \frac{1}{2}(\beta w^n\,,\,v)}_{\geq -\frac{1}{2}|||w^n|||\,|||\mathbf{a}\cdot\nabla w^n|||_*} + \underbrace{\frac{1}{2}(\mathbf{a}\cdot\nabla w^n\,,\,v)}_{=\frac{1}{2}|||\mathbf{a}\cdot\nabla w^n|||_*^2}
\end{aligned}$$

Estimation of the Convective Derivative I

- Assume that $|\mathbf{a}| \leq c_c\varepsilon$.
- Friedrichs' inequality implies $\left(\mathbf{a}\cdot\nabla(u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}})\,,\,v\right) \leq |\mathbf{a}|\left\|\nabla(u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}})\right\| c_\Omega \|\nabla v\|$.
- Hence $\left|\left|\left|\mathbf{a}\cdot\nabla(u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}})\right|\right|\right|_* \leq c_c c_\Omega \left|\left|\left|u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}}\right|\right|\right|$ and $\left|\left|\left|\mathbf{a}\cdot\nabla(u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}})\right|\right|\right|_*$ is equivalent to $\left|\left|\left|u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}}\right|\right|\right|$.

Estimation of the Convective Derivative II

- Assume that $|\mathbf{a}| \gg \varepsilon$.
- Auxiliary problem with analytical and discrete solutions Φ and $\Phi_{\mathcal{T}_n}$:

$$\varepsilon\,(\nabla\varphi\,,\,\nabla\psi) + \beta\,(\varphi\,,\,\psi) = \left(\mathbf{a}\cdot\nabla(u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}})\,,\,\psi\right) \qquad (*)$$

- $$\begin{aligned}\frac{1}{3}\left\{|||\Phi_{\mathcal{T}_n}||| + |||\Phi - \Phi_{\mathcal{T}_n}|||\right\} &\leq \left|\left|\left|\mathbf{a}\cdot\nabla(u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}})\right|\right|\right|_* \\ &\leq |||\Phi_{\mathcal{T}_n}||| + |||\Phi - \Phi_{\mathcal{T}_n}|||\end{aligned}$$
- $|||\Phi - \Phi_{\mathcal{T}_n}|||$ is equivalent to robust residual error indicator $\widetilde{\eta}^n_\tau$ for $(*)$.
- Hence $\left|\left|\left|\mathbf{a}\cdot\nabla(u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}})\right|\right|\right|_*$ is equivalent to $|||\Phi_{\mathcal{T}_n}||| + \widetilde{\eta}^n_\tau$.

Estimation of the Spatial Residual

- Spatial error indicator η^n_h:

$$\eta^n_h = \left\{\sum_{K\in\widetilde{\mathcal{T}}_n} {\alpha_K}^2 \|R_K\|_K^2 + \sum_{E\in\mathcal{E}_{\widetilde{\mathcal{T}}_n}} \varepsilon^{-\frac{1}{2}}\alpha_E \|R_E\|_E^2\right\}^{\frac{1}{2}}$$

- $\alpha_S = \min\left\{\varepsilon^{-\frac{1}{2}}h_S, \beta^{-\frac{1}{2}}\right\}$
- R_K and R_E are the usual element and interface residuals.
- Standard arguments for stationary problems yield: $|||R_h(u_{\mathcal{I}})|||_* \leq c^\dagger\eta^n_h + |||I^*_{\mathcal{M}}R_h(u_{\mathcal{I}})|||_*$, $\eta^n_h \leq c_\dagger\,|||R_h(u_{\mathcal{I}})|||_*$.
- $|||I^*_{\mathcal{M}}R_h(u_{\mathcal{I}})|||_*$ measures the consistency error of the stabilization.
- $c_\dagger$, $c^\dagger$ only depend on the polynomial degrees and on the shape parameters of the partitions $\widetilde{\mathcal{T}}_n$.

Proof of the Upper Bound

- L^2-representation:

$$\langle R_h(u_{\mathcal{I}})\,,\,v\rangle = \int_\Omega rv + \int_\Sigma jv$$

- Quasi-interpolation error estimate:

$$\|v - I_{\mathcal{M}}v\|_K \le c\alpha_K \,|||v|||_{\widetilde{\omega}_K}$$

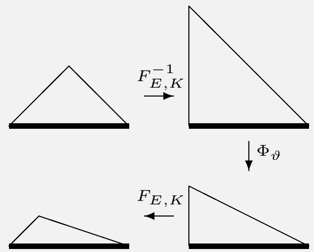
- Trace inequality:

$$\|v\|_E^2 \le \frac{|E|}{|K|}\,\|v\|_K^2 + \frac{2h_K\,|E|}{|K|}\,\|v\|_K\,\|\nabla v\|_K$$

Proof of the Lower Bound

- Insert $\psi_K R_K$ in L^2-representation with standard element cut-off functions ψ_K.
- Insert $\psi_{E,\vartheta} R_E$ in L^2-representation with squeezed face cut-off functions $\psi_{E,\vartheta}$ and

$$\vartheta = \varepsilon^{\frac{1}{2}} h_E^{-1} \alpha_E = \min\left\{1, \varepsilon^{\frac{1}{2}} h_E^{-1} \beta^{-\frac{1}{2}}\right\}.$$



Estimation of the consistency error $|||I_{\mathcal{M}}^* R_h(u_{\mathcal{I}})|||_*$

- $$|||I_{\mathcal{M}}^* R_h(u_{\mathcal{I}})|||_* = \sup_{v \in H_0^1(\Omega)} \frac{\langle R_h(u_{\mathcal{I}})\,,\,I_{\mathcal{M}}v\rangle}{|||v|||} = \sup_{v \in H_0^1(\Omega)} \frac{S_n(u_{\mathcal{I}}, I_{\mathcal{M}}v)}{|||v|||}$$
- Streamline diffusion and interior penalty methods:

$$|||I_{\mathcal{M}}^* R_h(u_{\mathcal{I}})|||_* \le c\eta_h^n$$

- Local projection scheme and subgrid-scale approach: $\nabla I_{\mathcal{M}}v \in \ker \kappa_{\mathcal{M}}$ and $I_{\mathcal{M}}v \in \ker \Pi_n$ hence

$$|||I_{\mathcal{M}}^* R_h(u_{\mathcal{I}})|||_* = 0.$$

A Posteriori Error Estimate

- Define the space-time error estimator by:

$$\eta^n = \tau_n^{\frac{1}{2}} \left[\underbrace{(\eta_h^n)^2}_{\text{spatial}} + \underbrace{\overbrace{\left|\left|\left| u_{\mathcal{T}_n}^n - u_{\mathcal{T}_{n-1}}^{n-1} \right|\right|\right|^2}^{=(\eta_\tau^n)^2} + \left(|||\Phi_{\mathcal{T}_n}||| + \widetilde{\eta}_\tau^n\right)^2}_{\text{temporal}} \right]^{\frac{1}{2}}.$$

- Then

$$\|e\|_{X(0,T)} \le c^* \left\{ \|u_0 - \pi_0 u_0\|^2 + \sum_{n=1}^{N_{\mathcal{I}}} (\eta^n)^2 \right\}^{\frac{1}{2}},$$

$$\eta^n \le c_* \|e\|_{X(t_{n-1},t_n)}.$$

- c_*, c^* only depend on the polynomial degrees and the shape parameters of the partitions $\widetilde{\mathcal{T}}_n$.

Overview

- The adaptive Algorithm yields a solution such that, for all times, its error is below a prescribed tolerance eps.
- The algorithm consists of several modules that coarsen and refine spatial meshes.
- An a priori bound for the energy of the discrete solution guarantees that the final time is actually reached within a finite number of time-steps.
- The coarsening of spatial meshes leads to an increase in the energy of the discrete solution which must effectively be controlled.
- Results hold for reaction-diffusion problems with dominant diffusion, ie. $\mathbf{a} = 0$ and $\beta \leq c_r \varepsilon$.

Adaptive Space-Time Algorithm

1. Given τ_0 and eps, choose ε_0, $\varepsilon_{h,\tau}$, ε_* such that $\varepsilon_0^2 + T\varepsilon_{h,\tau}^2 + \varepsilon_*^2 = \text{eps}^2$. Set $n = 0$.
2. Determine $\mathcal{T}_0$ and $\pi_0 u_0$ such that $\|u_0 - \pi_0 u_0\| \leq \varepsilon_0$.
3. Set $\tau_* = \varepsilon_*^2 / (\|f\|^2_{\Omega\times(0,T)} + \|\nabla(\pi_0 u_0)\|^2)$ minimal time-step.
4. Increment n by 1 and set $\tau_n = \min\{2\tau_{n-1}, T - t_{n-1}\}$.
5. Determine $\mathcal{T}_n = \texttt{COARSEN}(u^{n-1}_{\mathcal{T}_{n-1}}, \mathcal{T}_{n-1})$.
6. Determine $(u^n_{\mathcal{T}_n}, \tau_n, \mathcal{T}_n) = \texttt{ADAPT}(u^{n-1}_{\mathcal{T}_{n-1}}, \tau_n, \mathcal{T}_n, \mathcal{T}_{n-1}, \varepsilon_{h,\tau})$ and compute the energy-increment $\eta_{n,*} = \left|\!\left|\!\left|\pi_n u^{n-1}_{\mathcal{T}_{n-1}}\right|\!\right|\!\right|^2 - \left|\!\left|\!\left|u^{n-1}_{\mathcal{T}_{n-1}}\right|\!\right|\!\right|^2 - \frac{1}{\tau_n}\left\|u^n_{\mathcal{T}_n} - \pi_n u^{n-1}_{\mathcal{T}_{n-1}}\right\|^2$.
7. If $\eta_{n,*} \leq 0$, go to 8. Otherwise determine $\mathcal{T}_n = \texttt{REFINE}(\eta_{n,*}, \mathcal{T}_n, \mathcal{T}_{n-1})$ and go to 6.
8. If $t_{n-1} + \tau_n = T$, stop. Otherwise go to 4.

Module COARSEN

- $\texttt{COARSEN}(u^{n-1}_{\mathcal{T}_{n-1}}, \mathcal{T}_{n-1})$ produces a partial coarsening of $\mathcal{T}_{n-1}$.
- This module may not be based on an error indicator at all; its output may be independent of $u^{n-1}_{\mathcal{T}_{n-1}}$ and may equal $\mathcal{T}_{n-1}$.
- This module should remove as many degrees of freedom as possible while keeping the difference of $u^{n-1}_{\mathcal{T}_{n-1}}$ to a suitable interpolation in the resulting finite element space at a moderate size.

Module REFINE

- This module is used in two different ways depending on its input arguments.
- Given a partition $\mathcal{T}$ and an error indicator $\eta = (\sum_K \eta_K^2)^{\frac{1}{2}}$, $\texttt{REFINE}(\eta, \mathcal{T})$ produces a new admissible partition such that at least one element in the subset $\operatorname{argmax}_{K\in\mathcal{T}} \eta_K$ of $\mathcal{T}$ is refined.
- Given a partition $\mathcal{T}$, an associated error indicator η and a second partition $\mathcal{T}'$, $\texttt{REFINE}(\eta, \mathcal{T}, \mathcal{T}')$ has the same effect as $\texttt{REFINE}(\eta, \mathcal{T})$ with the additional condition that at least one element of $\mathcal{T} \setminus \mathcal{T}'$ which has previously been coarsened is refined.

Module ADAPT

1. Solve the discrete problem for $\mathcal{T}_n$ and τ_n; compute η_τ^n, η_h^n.
2. If $(\eta_\tau^n)^2 + (\eta_h^n)^2 \le \varepsilon_{h,\tau}^2$, stop; otherwise go to 3.
3. If $\eta_h^n > \eta_\tau^n$, set $\mathcal{T}_n = \mathtt{REFINE}(\eta_h^n, \mathcal{T}_n)$ and go to 1; else set $\eta_1^n = \left|\left|\left| u_{\mathcal{T}_n}^n - \pi_n u_{\mathcal{T}_{n-1}}^{n-1} \right|\right|\right|$ and $\eta_2^n = \left|\left|\left| u_{\mathcal{T}_{n-1}}^{n-1} - \pi_n u_{\mathcal{T}_{n-1}}^{n-1} \right|\right|\right|$.
4. If $\tau_n > \tau_*$ go to 5; else go to 6.
5. If $\eta_1^n > \eta_2^n$, set $\tau_n = \max\{\frac{1}{2}\tau_n, \tau_*\}$; else set $\mathcal{T}_n = \mathtt{REFINE}(\eta_h^n, \mathcal{T}_n, \mathcal{T}_{n-1})$ and go to 1.
6. If $(\eta_h^n)^2 + 2(\eta_2^n)^2 \le \varepsilon_{h,\tau}^2$, stop; else go to 7.
7. If $\eta_h^n > \sqrt{2}\eta_2^n$, set $\mathcal{T}_n = \mathtt{REFINE}(\eta_h^n, \mathcal{T}_n)$; else set $\mathcal{T}_n = \mathtt{REFINE}(\eta_h^n, \mathcal{T}_n, \mathcal{T}_{n-1})$. Go to 1.

Convergence Proof

- Difficulty: The goal $\eta \le \mathrm{eps}$ involves all times while, at any intermediate time, only information up to that time is available.
- Basic steps:
  - Prove an a priori bound for the discrete energy depending only on the given data.
  - Using convergence results for stationary problems, prove the termination of the module REFINE.
  - Prove the termination of the module ADAPT.

Modifications and Open Problems

- Instead on parabolic energy estimates, one can also base the a posteriori error analysis on properties of the evolution operator and elliptic reconstruction. This is an alternative way to decouple the temporal and spatial error.
- For the spatial error one may also use other a posteriori error indicators, e.g. auxiliary local discrete problems, $H(\mathrm{div})$-liftings, ... These, however, may not be robust.
- Mildly nonlinear problems may be handled similarly. The results, however, are less complete and are based on the assumption that the variational solution is a regular one in the sense of the implicit function theorem.
- Contrary to stationary problems, for time-dependent problems, the optimality of the adaptive algorithm is completely open.

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