A Posteriori Error Analysis of the Method of Characteristics

C. Bernardi F. Hecht R. Verfürth

Université Pierre et Marie Curie, Laboratoire Jacques-Louis Lions

Ruhr-Universität Bochum, Fakultät für Mathematik

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Differential Equation

$$\begin{aligned} \partial_t u - \operatorname{div}(d\nabla u) + \mathbf{a}\cdot\nabla u + ru &= f && \text{in } \Omega\times(0,T] \\ u &= 0 && \text{on } \Gamma\times(0,T] \\ u &= u_0 && \text{in } \Omega \end{aligned}$$

- $d > 0$
- $r \geq 0$
- $\mathbf{a} \in C^1(\Omega\times(0,T])^d$
- $\operatorname{div}\mathbf{a} = 0$ in $\Omega\times(0,T]$
- $\mathbf{a} = 0$ on $\Gamma\times(0,T]$

Norms

- Energy norm

$$|||v||| = \left\{d\|\nabla v\|^2 + r\|v\|^2\right\}^{\frac{1}{2}}$$

- Dual norm

$$|||\varphi|||_* = \sup_{v\in H_0^1(\Omega)\setminus\{0\}} \frac{\langle\varphi, v\rangle}{|||v|||}$$

- Error norm

$$\|u\|_{X(a,b)} = \Big\{\operatorname*{ess.sup}_{t\in(a,b)}\|u(\cdot,t)\|^2 + \int_a^b |||u(\cdot,t)|||^2 dt + \int_a^b |||(\partial_t u + \mathbf{a}\cdot\nabla u)(\cdot,t)|||_*^2 dt\Big\}^{\frac{1}{2}}$$

Method of Characteristics

- For every $(x^*, t^*) \in \Omega\times(0,T]$ the characteristic equation

$$\begin{aligned} \frac{d}{dt}x(t;x^*,t^*) &= \mathbf{a}(x(t;x^*,t^*),t), \qquad t\in(0,t^*), \\ x(t^*;x^*,t^*) &= x^* \end{aligned}$$

  admits a unique global solution.
- Set $U(x^*,t) = u(x(t;x^*,t^*),t)$.
- Then

$$d_t U - \operatorname{div}(d\nabla u) + ru = f \quad \text{in } \Omega\times(0,T).$$

- Discretize $d_t U$ by backward differences and spatial derivatives by standard finite elements.

Meshes and Spaces

- $\mathcal{I} = \{(t_{n-1}, t_n] : 1 \le n \le N_{\mathcal{I}}\}$ partition of $[0, T]$.
- $\tau_n = t_n - t_{n-1}$.
- $\mathcal{T}_n$, $0 \le n \le N_{\mathcal{I}}$, affine equivalent, admissible, shape regular partitions of Ω.
- Transition condition: There is a common refinement $\widetilde{\mathcal{T}}_n$ of $\mathcal{T}_n$ and $\mathcal{T}_{n-1}$ such that $h_K \le c h_{K'}$ for all $K \in \mathcal{T}_n$ and all $K' \in \widetilde{\mathcal{T}}_n$ with $K' \subset K$.
- $V_n \subset H_0^1(\Omega)$ finite element space corresponding to $\mathcal{T}_n$.
- Lagrange condition: Functions in V_n are uniquely determined by their values in a set $\mathcal{V}_n$ of nodes.

Discrete Problem

- Set $u_{\mathcal{T}_0}^0 = \pi_0 u_0$.
- For $n = 1, \ldots, N_{\mathcal{I}}$:
  - Compute an approximation x_z^{n-1} to $x(t_{n-1}; z, t_n)$ for every $z \in \mathcal{V}_{n,\Omega}$.
  - Compute $\widetilde{u}_{\mathcal{T}_n}^{n-1} \in V_n$ such that

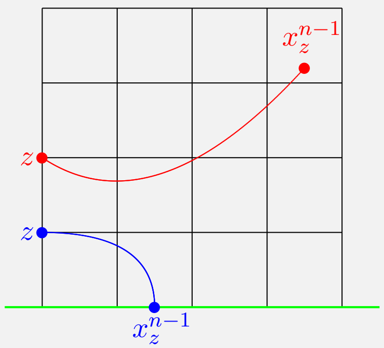
$$\widetilde{u}_{\mathcal{T}_n}^{n-1}(z) = \begin{cases} u_{\mathcal{T}_{n-1}}^{n-1}(x_z^{n-1}) & \text{if } z \in \mathcal{V}_{n,\Omega}, \\ 0 & \text{if } z \in \mathcal{V}_{n,\Gamma}. \end{cases}$$

  - Find $u_{\mathcal{T}_n}^n \in V_n$ such that

$$\Big(\frac{u_{\mathcal{T}_n}^n - \widetilde{u}_{\mathcal{T}_n}^{n-1}}{\tau_n}, v_{\mathcal{T}_n}\Big) + (d\nabla u_{\mathcal{T}_n}^n, \nabla v_{\mathcal{T}_n}) + (r u_{\mathcal{T}_n}^n, v_{\mathcal{T}_n}) = (f, v_{\mathcal{T}_n})$$

  holds for all $v_{\mathcal{T}_n} \in V_n$.

Re-interpolation



Basic Steps

- Error and residual are equivalent.
- The residual splits into a spatial and a temporal residual. The norm of the sum of these is equivalent to the sum of their norms.
- Derive a reliable, efficient and robust error indicator for the temporal residual.
- Derive a reliable, efficient and robust error indicator for the spatial residual.

Equivalence of Error and Residual

- $u_{\mathcal{I}}$ continuous piece-wise affine, equals $u^n_{\mathcal{I}_n}$ at t_n.
- Residual:

$$\langle R(u_{\mathcal{I}}), v\rangle = (f, v) - (\partial_t u_{\mathcal{I}}, v) - (d\nabla u_{\mathcal{I}}, \nabla v) - (\mathbf{a}\cdot\nabla u_{\mathcal{I}}, v) - (ru_{\mathcal{I}}, v)$$

- Lower bound:

$$\|R(u_{\mathcal{I}})\|_{L^2(t_{n-1},t_n;H^{-1}(\Omega))} \le \sqrt{2}\|u - u_{\mathcal{I}}\|_{X(t_{n-1},t_n)}$$

- Upper bound:

$$\|u - u_{\mathcal{I}}\|_{X(0,t_n)} \le \left\{4\|u_0 - \pi_0 u_0\|^2 + 6\|R(u_{\mathcal{I}})\|^2_{L^2(0,t_n;H^{-1}(\Omega))}\right\}^{\frac{1}{2}}$$

Proof of the Equivalence

- Relation of residual and error:

$$\langle R(u_{\mathcal{I}}), v\rangle = (\partial_t e, v) - (\mathbf{a}\cdot\nabla e, v) - (d\nabla e, \nabla v) - (re, v)$$

- Lower bound: Definition of primal and dual norm plus Cauchy-Schwarz inequality.
- Upper bound: Parabolic energy estimate with $v = e$ as test-function.

Decomposition of the Residual

- Temporal residual:

$$\langle R_\tau(u_{\mathcal{I}}), v\rangle = (d\nabla(u^n_{\mathcal{I}_n} - u_{\mathcal{I}}), \nabla v) + (\mathbf{a}\cdot\nabla(u^n_{\mathcal{I}_n} - u_{\mathcal{I}}), v) + (r(u^n_{\mathcal{I}_n} - u_{\mathcal{I}}), v)$$

- Spatial residual:

$$\langle R_h(u_{\mathcal{I}}), v\rangle = (f, v) - (\partial_t u_{\mathcal{I}}, v) - (d\nabla u^n_{\mathcal{I}_n}, \nabla v) - (\mathbf{a}\cdot\nabla u^n_{\mathcal{I}_n}, v) - (ru^n_{\mathcal{I}_n}, v)$$

- Splitting: $R(u_{\mathcal{I}}) = R_\tau(u_{\mathcal{I}}) + R_h(u_{\mathcal{I}})$
- Estimate for $L^2(t_{n-1}, t_n; H^{-1}(\Omega))$-norms:

$$\frac{1}{5}\left\{\|R_\tau(u_{\mathcal{I}})\|^2 + \|R_h(u_{\mathcal{I}})\|^2\right\}^{\frac{1}{2}} \le \|R_\tau(u_{\mathcal{I}}) + R_h(u_{\mathcal{I}})\| \le \|R_\tau(u_{\mathcal{I}})\| + \|R_h(u_{\mathcal{I}})\|$$

Motivation of the Lower Bound

- Strengthened Cauchy-Schwarz inequality for $v = c$ and $w = \frac{b-t}{b-a}$:

$$\int_a^b vw = \frac{1}{2}c(b-a) = \frac{\sqrt{3}}{2}\|v\|_{(a,b)}\|w\|_{(a,b)}$$

- Hence:

$$\|v + w\|^2_{(a,b)} \ge \left(1 - \frac{\sqrt{3}}{2}\right)\left\{\|v\|^2_{(a,b)} + \|w\|^2_{(a,b)}\right\}$$

Proof of the Lower Bound

- $R_h(u_{\mathcal{I}})$ is piece-wise constant.
- $R_\tau(u_{\mathcal{I}})$ is piece-wise affine: $R_\tau(u_{\mathcal{I}}) = \frac{t_n - t}{\tau_n}\rho^n$ with

$$\langle \rho^n, v\rangle = (d\nabla(u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}}), \nabla v) + (\mathbf{a}\cdot\nabla(u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}}), v) + (r(u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}}), v).$$

- Choose $v, w \in H^1_0(\Omega)$ such that

$$|||v||| = |||R_h(u_{\mathcal{I}})|||_*, \qquad \langle R_h(u_{\mathcal{I}}), v\rangle = |||R_h(u_{\mathcal{I}})|||_*^2,$$
$$|||w||| = |||\rho^n|||_*, \qquad \langle \rho^n, w\rangle = |||\rho^n|||_*^2.$$

- Insert $3\left(\frac{t-t_{n-1}}{\tau_n}\right)^2 v + \frac{t_n - t}{\tau_n} w$ as test-function in representation of $R(u_{\mathcal{I}})$.

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Estimation of the Temporal Residual

- Recall $R_\tau(u_{\mathcal{I}}) = \frac{t_n - t}{\tau_n}\rho^n$ with

$$\langle \rho^n, v\rangle = (d\nabla(u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}}), \nabla v) + (\mathbf{a}\cdot\nabla(u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}}), v) + (r(u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}}), v).$$

- Upper bound:

$$|||\rho^n|||_* \le \left\{|||u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}}||| + |||\mathbf{a}\cdot\nabla(u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}})|||_*\right\}$$

- Follows from definition of ρ^n and $|||\cdot|||_*$.
- Lower bound:

$$\frac{1}{3}\left\{|||u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}}||| + |||\mathbf{a}\cdot\nabla(u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}})|||_*\right\} \le |||\rho^n|||_*$$

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Proof of the Lower Bound

- Set $w^n = u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}}$ and choose $v \in H^1_0(\Omega)$ with $|||v||| = |||\mathbf{a}\cdot\nabla w^n|||_*$ and $(\mathbf{a}\cdot\nabla w^n, v) = |||\mathbf{a}\cdot\nabla w^n|||_*^2$
- Insert $\frac{1}{2}w^n + \frac{1}{2}v$ in the definition of ρ^n:

$$\begin{aligned}&\langle \rho^n, \frac{1}{2}w^n + \frac{1}{2}v\rangle \\ &= \underbrace{\frac{1}{2}(d\nabla w^n, \nabla w^n) + \frac{1}{2}(rw^n, w^n)}_{=\frac{1}{2}|||w^n|||^2} + \underbrace{\frac{1}{2}(\mathbf{a}\cdot\nabla w^n, w^n)}_{=0} \\ &+ \underbrace{\frac{1}{2}(d\nabla w^n, \nabla v) + \frac{1}{2}(rw^n, v)}_{\ge -\frac{1}{2}|||w^n||| \, |||\mathbf{a}\cdot\nabla w^n|||_*} + \underbrace{\frac{1}{2}(\mathbf{a}\cdot\nabla w^n, v)}_{=\frac{1}{2}|||\mathbf{a}\cdot\nabla w^n|||_*^2}\end{aligned}$$

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Estimation of the Convective Derivative I.

- Assume that $\|\mathbf{a}\|_\infty \le c_c d$.
- Friedrichs' inequality implies

$$\begin{aligned}(\mathbf{a}\cdot\nabla(u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}}), v) &\le \|\mathbf{a}\|_\infty \|\nabla(u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}})\| \|v\| \\ &\le \|\mathbf{a}\|_\infty \|\nabla(u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}})\| c_\Omega \|\nabla v\|\end{aligned}$$

- Hence $|||\mathbf{a}\cdot\nabla(u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}})|||_* \le c_c c_\Omega |||u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}}|||$ and $|||\mathbf{a}\cdot\nabla(u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}})|||_*$ is equivalent to $|||u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}}|||$.

Estimation of the Convective Derivative II.

- Assume that $\|\mathbf{a}\|_\infty \gg d$.
- Consider the auxiliary problem

$$d(\nabla v^n_{\mathcal{T}_n}, \nabla\varphi) + r(v^n_{\mathcal{T}_n}, \varphi) = (\mathbf{a}\cdot\nabla(u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}}), \varphi) \qquad (*)$$

  with variational and discrete solutions Φ and $\varphi_{\mathcal{T}_n}$.
- Then $|||\Phi||| = |||\mathbf{a}\cdot\nabla(u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}})|||_*$, $|||\varphi_{\mathcal{T}_n}||| \le |||\mathbf{a}\cdot\nabla(u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}})|||_*$ and therefore

$$\begin{aligned}\frac{1}{3}\{|||\varphi_{\mathcal{T}_n}||| + |||\Phi - \varphi_{\mathcal{T}_n}|||\} &\le |||\mathbf{a}\cdot\nabla(u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}})|||_* \\ &\le |||\varphi_{\mathcal{T}_n}||| + |||\Phi - \varphi_{\mathcal{T}_n}|||.\end{aligned}$$

- Transition condition implies that $|||\Phi - \varphi_{\mathcal{T}_n}|||$ is equivalent to every robust, i.e. residual, error indicator η^n_τ for $(*)$.
- Hence $|||\mathbf{a}\cdot\nabla(u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}})|||_*$ is equivalent to $|||\varphi_{\mathcal{T}_n}||| + \eta^n_\tau$.

Estimation of the Spatial Residual

- $R_h(u_{\mathcal{I}})$ does not satisfy the Galerkin orthogonality, but

$$\langle R_h(u_{\mathcal{I}}), v_{\mathcal{T}_n}\rangle = \Big(\frac{u^{n-1}_{\mathcal{T}_n} - \widetilde{u}^{n-1}_{\mathcal{T}_n}}{\tau_n} - \mathbf{a}\cdot\nabla u^n_{\mathcal{T}_n}, v_{\mathcal{T}_n}\Big).$$

- Consider the auxiliary problem

$$d(\nabla w^n_{\mathcal{T}_n}, \nabla\varphi) + r(w^n_{\mathcal{T}_n}, \varphi) = \Big(\frac{u^{n-1}_{\mathcal{T}_n} - \widetilde{u}^{n-1}_{\mathcal{T}_n}}{\tau_n} - \mathbf{a}\cdot\nabla u^n_{\mathcal{T}_n}, \varphi\Big) \quad (\sharp)$$

  with variational and discrete solutions Ψ and $\psi_{\mathcal{T}_n}$.
- Then $|||\Psi||| = |||R_h(u_{\mathcal{I}})|||_*$, $|||\psi_{\mathcal{T}_n}||| \le |||R_h(u_{\mathcal{I}})|||_*$ and $\frac{1}{3}\{|||\psi_{\mathcal{T}_n}||| + |||\Psi - \psi_{\mathcal{T}_n}|||\} \le |||R_h(u_{\mathcal{I}})|||_* \le |||\psi_{\mathcal{T}_n}||| + |||\Psi - \psi_{\mathcal{T}_n}|||$.
- Transition condition implies that $|||\Psi - \psi_{\mathcal{T}_n}|||$ is equivalent to every robust, i.e. residual, error indicator η^n_h for $(\sharp)$.
- Hence $|||R_h(u_{\mathcal{I}})|||_*$ is equivalent to $|||\psi_{\mathcal{T}_n}||| + \eta^n_h$.

Computation of η^n_h

- Want to replace $\widetilde{u}^{n-1}_{\mathcal{T}_n}$ by $u^{n-1}_{\mathcal{T}_{n-1}}$.
- Integration by parts element-wise yields:

$$\begin{aligned}&\langle R_h(u_{\mathcal{I}}),\varphi\rangle - d(\nabla w^n_{\mathcal{T}_n}, \nabla\varphi) - r(w^n_{\mathcal{T}_n}, \varphi) \\ &= \sum_{K\in\mathcal{T}_n}\int_K \Big(f - \frac{u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}}}{\tau_n} + \operatorname{div}(d\nabla u^n_{\mathcal{T}_n}) \\ &\qquad - \mathbf{a}\cdot\nabla u^n_{\mathcal{T}_n} - r u^n_{\mathcal{T}_n} + d\Delta w^n_{\mathcal{T}_n} - r w^n_{\mathcal{T}_n}\Big)\varphi \\ &\quad + \sum_{E\in\mathcal{E}_n}\int_E \mathbb{J}_E(\mathbf{n}_E\cdot(d\nabla u^n_{\mathcal{T}_n}) - d\mathbf{n}_E\cdot\nabla w^n_{\mathcal{T}_n})\varphi.\end{aligned}$$

- η^n_h consists of element and face residuals with weighting factors $\min\{d^{-\frac{1}{2}}h_K, r^{-\frac{1}{2}}\}$ and $d^{-\frac{1}{4}}\min\{d^{-\frac{1}{2}}h_K, r^{-\frac{1}{2}}\}^{\frac{1}{2}}$.

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