Computational Fluid Dynamics

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Fundamentals

- Modelization
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Deformation of Materials

- Notation:
  - $\Omega \subset \mathbb{R}^d$: domain initially occupied by a material moving under the influence of interior and exterior forces
  - $\eta \in \Omega$: initial position of an arbitrary particle
  - $x = \Phi(\eta, t)$: position of particle η at time $t > 0$
  - $\Omega(t) = \Phi(\Omega, t)$: domain occupied by the material at time $t > 0$
- Basic assumptions:
  - $\Phi(\cdot, t) : \Omega \to \Omega(t)$ is an orientation preserving diffeomorphism for all $t > 0$.
  - $\Phi(\cdot, 0)$ is the identity.

Lagrange and Euler Representation

- Lagrange representation: Fix η and look at the trajectory $t \mapsto \Phi(\eta, t)$. η is called Lagrange coordinate. The Langrange coordinate system moves with the fluid.
- Euler representation: Fix the point x and look at the trajectory $t \mapsto \Phi(\cdot, t)^{-1}(x)$ which passes through x. x is called Euler coordinate. The Euler coordinate system is fixed.

Velocity

Velocity of the movement at the point $x = \Phi(\eta, t)$ is

$$\mathbf{v}(x,t) = \frac{\partial}{\partial t}\Phi(\eta, t).$$

Properties

$D\Phi = (\frac{\partial \Phi_i}{\partial \eta_j})_{1\le i,j\le d}$ Jacobi matrix of Φ, $J = \det D\Phi$ Jacobi determinant of Φ, A_{ij} co-factors of $D\Phi$ $(1 \le i,j \le d)$:

$$\begin{aligned}
\frac{\partial}{\partial t}J &= \sum_{i,j} \frac{\partial}{\partial (D\Phi)_{ij}} J \frac{\partial}{\partial t}(D\Phi)_{ij} &&= \sum_{i,j} (-1)^{i+j} A_{ij} \frac{\partial^2}{\partial t \partial \eta_j}\Phi_i \\
&= \sum_{i,j} (-1)^{i+j} A_{ij} \frac{\partial}{\partial \eta_j}\mathbf{v}_i &&= \sum_{i,j,k} (-1)^{i+j} A_{ij} \frac{\partial}{\partial \eta_j}\Phi_k \frac{\partial}{\partial x_k}\mathbf{v}_i \\
&= \sum_{i,k} J\delta_{i,k} \frac{\partial}{\partial x_k}\mathbf{v}_i &&= J \operatorname{div} \mathbf{v}
\end{aligned}$$

Transport Theorem

$$\begin{aligned}
&\frac{d}{dt}\int_{V(t)} f(x,t)dx \\
&= \frac{d}{dt}\int_V f(\Phi(\eta,t),t)J(\eta,t)d\eta \\
&= \int_V \Big(\frac{\partial}{\partial t} f(\Phi(\eta,t),t)J(\eta,t) \\
&\qquad + \nabla f(\Phi(\eta,t),t)\cdot \mathbf{v}(\Phi(\eta,t),t)J(\eta,t) \\
&\qquad + f(\Phi(\eta,t),t) \operatorname{div} \mathbf{v}(\Phi(\eta,t),t)J(\eta,t)\Big)d\eta \\
&= \int_{V(t)} \Big(\frac{\partial}{\partial t} f(x,t) + \operatorname{div}\big[f(x,t)\mathbf{v}(x,t)\big]\Big)dx
\end{aligned}$$

Conservation of Mass

- ρ denotes the density of the material.
- $\int_{V(t)} \rho dx$ is the total mass of a control volume.
- Total mass is conserved:

$$0 = \frac{d}{dt}\int_{V(t)} \rho dx = \int_{V(t)} \left(\frac{\partial}{\partial t}\rho + \operatorname{div}\left[\rho \mathbf{v}\right]\right) dx.$$

- This holds for every control volume, hence:

$$\frac{\partial}{\partial t}\rho + \operatorname{div}\left[\rho \mathbf{v}\right] = 0.$$

Conservation of Momentum

- $\int_{V(t)} \rho \mathbf{v} dx$ is the total momentum of a control volume.
- Its temporal change is

$$\frac{d}{dt}\int_{V(t)} \rho \mathbf{v} dx = \int_{V(t)} \left(\frac{\partial}{\partial t}\left[\rho \mathbf{v}\right] + \operatorname{div}\left[\rho \mathbf{v} \otimes \mathbf{v}\right]\right) dx.$$

- This is in equilibrium with exterior and interior forces.
- Exterior forces are given by $\int_{V(t)} \rho \mathbf{f} dx$.

Interior Forces

Basic assumptions:

- Interior forces act via the surface of a volume $V(t)$.
- Interior forces only depend on the normal direction of the surface of the volume.
- Interior forces are additive and continuous.

Cauchy Theorem

The previous assumptions imply:

- There is a tensor field $\underline{\mathbf{T}} : \Omega \to \mathbb{R}^{d \times d}$ such that the interior forces are given by $\int_{\partial V(t)} \underline{\mathbf{T}} \cdot \mathbf{n} dS$.
- $\underline{\mathbf{T}}$ is such that the divergence theorem of Gauß holds

$$\int_{\partial V(t)} \underline{\mathbf{T}} \cdot \mathbf{n} dS = \int_{V(t)} \operatorname{div} \underline{\mathbf{T}} dx.$$

Conservation of Momentum (ctd.)

- The conservation of momentum and the Cauchy theorem imply:

$$\int_{V(t)} \Big(\frac{\partial}{\partial t}(\rho \mathbf{v}) + \operatorname{div}(\rho \mathbf{v} \otimes \mathbf{v}) \Big) = \int_{V(t)} \Big(\rho \mathbf{f} + \operatorname{div} \underline{\mathbf{T}} \Big).$$

- This holds for every control volume, hence:

$$\frac{\partial}{\partial t}(\rho \mathbf{v}) + \operatorname{div}(\rho \mathbf{v} \otimes \mathbf{v}) = \rho \mathbf{f} + \operatorname{div} \underline{\mathbf{T}}.$$

Conservation of Energy

- $\int_{V(t)} e dx$ is the total energy of a control volume.
- Its temporal change is in equilibrium with the internal energy and the energy of exterior and interior forces.
- Exterior forces contribute $\int_{V(t)} \rho \mathbf{f} \cdot \mathbf{v} dx$.
- Interior forces give $\int_{\partial V(t)} \mathbf{n} \cdot \underline{\mathbf{T}} \cdot \mathbf{v} dS = \int_{V(t)} \operatorname{div} \big[\underline{\mathbf{T}} \cdot \mathbf{v} \big] dx$.
- The Cauchy theorem implies that the internal energy is of the form $\int_{\partial V(t)} \mathbf{n} \cdot \boldsymbol{\sigma} dS = \int_{V(t)} \operatorname{div} \boldsymbol{\sigma} dx$.
- Hence, conservation of energy implies

$$\frac{\partial}{\partial t} e + \operatorname{div}(e \mathbf{v}) = \rho \mathbf{f} \cdot \mathbf{v} + \operatorname{div}(\underline{\mathbf{T}} \cdot \mathbf{v}) + \operatorname{div} \boldsymbol{\sigma}.$$

Constitutive Laws

Basic assumptions:

- $\underline{\mathbf{T}}$ only depends on the gradient of the velocity.
- The dependence on the velocity gradient is linear.
- $\underline{\mathbf{T}}$ is symmetric.
  (Due to the Cauchy theorem this is a consequence of the conservation of angular momentum.)
- In the absence of internal friction, $\underline{\mathbf{T}}$ is diagonal and proportional to the pressure, i.e. all interior forces act in normal direction.
- The total energy e is the sum of internal and kinetic energy.
- $\boldsymbol{\sigma}$ is proportional to the variation of the internal energy.

Consequences of the Constitutive Laws

Above assumptions imply:

- $\underline{\mathbf{T}} = 2\lambda \underline{\mathbf{D}}(\mathbf{v}) + \mu(\operatorname{div} \mathbf{v}) \underline{\mathbf{I}} - p \underline{\mathbf{I}}$,
  where $\underline{\mathbf{D}}(\mathbf{v}) = \frac{1}{2}(\nabla \mathbf{v} + \nabla \mathbf{v}^t)$ is the deformation tensor, λ, μ are the dynamic viscosities, p is the pressure, $\underline{\mathbf{I}}$ is the unit tensor.
- $e = \rho \varepsilon + \frac{1}{2} \rho |\mathbf{v}|^2$,
  where ε is often identified with the temperature.
- $\boldsymbol{\sigma} = \alpha \nabla \varepsilon$.

Compressible Navier-Stokes Equations in Conservative Form

$$
\begin{aligned}
\frac{\partial}{\partial t}\rho + \operatorname{div}(\rho \mathbf{v}) &= 0 \\
\frac{\partial}{\partial t}(\rho \mathbf{v}) + \operatorname{div}(\rho \mathbf{v} \otimes \mathbf{v}) &= \rho \mathbf{f} + 2\lambda \operatorname{div} \underline{\mathbf{D}}(\mathbf{v}) \\
&\quad + \mu \operatorname{grad} \operatorname{div} \mathbf{v} - \operatorname{grad} p \\
\frac{\partial}{\partial t} e + \operatorname{div}(e\mathbf{v}) &= \rho \mathbf{f} \cdot \mathbf{v} + 2\lambda \operatorname{div}[\underline{\mathbf{D}}(\mathbf{v}) \cdot \mathbf{v}] \\
&\quad + \mu \operatorname{div}[\operatorname{div} \mathbf{v} \cdot \mathbf{v}] - \operatorname{div}(p\mathbf{v}) + \alpha \Delta \varepsilon \\
p &= p(\rho, \varepsilon) \\
e &= \rho\varepsilon + \frac{1}{2}\rho |\mathbf{v}|^2
\end{aligned}
$$

Euler Equations

Inviscid flows, i.e. $\lambda = \mu = 0$:

$$
\begin{aligned}
\frac{\partial}{\partial t}\rho + \operatorname{div}(\rho \mathbf{v}) &= 0 \\
\frac{\partial}{\partial t}(\rho \mathbf{v}) + \operatorname{div}(\rho \mathbf{v} \otimes \mathbf{v} + p\underline{\mathbf{I}}) &= \rho \mathbf{f} \\
\frac{\partial}{\partial t} e + \operatorname{div}(e\mathbf{v} + p\mathbf{v}) &= \rho \mathbf{f} \cdot \mathbf{v} + \alpha \Delta \varepsilon \\
p &= p(\rho, \varepsilon) \\
e &= \rho\varepsilon + \frac{1}{2}\rho |\mathbf{v}|^2
\end{aligned}
$$

Compressible Navier-Stokes Equations in Non-Conservative Form

Insert first equation in second one and first and second equation in third one:

$$
\begin{aligned}
\frac{\partial}{\partial t}\rho + \operatorname{div}(\rho \mathbf{v}) &= 0 \\
\rho[\frac{\partial}{\partial t}\mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v}] &= \rho \mathbf{f} + \lambda \Delta \mathbf{v} + (\lambda + \mu) \operatorname{grad} \operatorname{div} \mathbf{v} - \operatorname{grad} p \\
\rho[\frac{\partial}{\partial t}\varepsilon + \rho \mathbf{v} \cdot \operatorname{grad} \varepsilon] &= \lambda \underline{\mathbf{D}}(\mathbf{v}) : \underline{\mathbf{D}}(\mathbf{v}) + \mu (\operatorname{div} \mathbf{v})^2 - p \operatorname{div} \mathbf{v} \\
&\quad + \alpha \Delta \varepsilon \\
p &= p(\rho, \varepsilon)
\end{aligned}
$$

Non-Stationary Incompressible Navier-Stokes Equations

- Assume that the density ρ is constant,
- replace p by $\frac{p}{\rho}$,
- denote by $\nu = \frac{\lambda}{\rho}$ the kinematic viscosity,
- drop the energy equation:

$$
\begin{aligned}
\operatorname{div} \mathbf{v} &= 0 \\
\frac{\partial}{\partial t}\mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} &= \mathbf{f} + \nu \Delta \mathbf{v} - \operatorname{grad} p
\end{aligned}
$$

Reynolds' Number

- Introduce a reference length L, a reference time T, a reference velocity U, a reference pressure P, and a reference force F and new variables and quantities by $x = Ly$, $t = T\tau$, $\mathbf{v} = U\mathbf{u}$, $p = Pq$, $\mathbf{f} = F\mathbf{g}$.
- Choose T, F and P such that $T = \frac{L}{U}$, $F = \frac{\nu U}{L^2}$ and $\frac{PL}{\nu U} = 1$.
- Then

$$\operatorname{div} \mathbf{u} = 0$$
$$\frac{\partial}{\partial t}\mathbf{u} + Re(\mathbf{u} \cdot \nabla)\mathbf{u} = \mathbf{f} + \Delta\mathbf{u} - \operatorname{grad} q,$$

where $Re = \frac{LU}{\nu}$ is the dimensionless Reynolds' number.

Stationary Incompressible Navier-Stokes Equations

Assume that the flow is stationary:

$$\operatorname{div} \mathbf{v} = 0$$
$$-\nu\Delta\mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} + \operatorname{grad} p = \mathbf{f}$$

Stokes Equations

Linearize at velocity $\mathbf{v} = 0$:

$$\operatorname{div} \mathbf{v} = 0$$
$$-\Delta\mathbf{v} + \operatorname{grad} p = \mathbf{f}$$

Boundary Conditions

- Around 1827, Pierre Louis Marie Henri Navier suggested the general boundary condition

$$\lambda_n \mathbf{v} \cdot \mathbf{n} + (1 - \lambda_n)\mathbf{n} \cdot \underline{\mathbf{T}} \cdot \mathbf{n} = 0$$
$$\lambda_t[\mathbf{v} - (\mathbf{v} \cdot \mathbf{n})\mathbf{n}] + (1 - \lambda_t)[\underline{\mathbf{T}} \cdot \mathbf{n} - (\mathbf{n} \cdot \underline{\mathbf{T}} \cdot \mathbf{n})\mathbf{n}] = 0$$

  with parameters $\lambda_n, \lambda_t \in [0, 1]$ depending on the actual flow-problem.
- A particular case is the slip boundary condition $\mathbf{v} \cdot \mathbf{n} = 0$, $\underline{\mathbf{T}} \cdot \mathbf{n} - (\mathbf{n} \cdot \underline{\mathbf{T}} \cdot \mathbf{n})\mathbf{n} = 0$.
- Around 1845, Sir George Gabriel Stokes suggested the no-slip boundary condition $\mathbf{v} = 0$.

Sobolev Spaces and Norms

- $L^2(\Omega)$ Lebesgue space with norm $\|\varphi\|_\Omega = \|\varphi\| = \left\{\int_\Omega \varphi^2\right\}^{\frac{1}{2}}$
- $H^k(\Omega) = \{\varphi \in L^2(\Omega) : D^\alpha\varphi \in L^2(\Omega)\,\forall \alpha_1 + \ldots + \alpha_d \le k\}$, $k \ge 1$, Sobolev spaces with semi-norm
  $|\varphi|_{k,\Omega} = |\varphi|_k = \left\{\sum_{\alpha_1+\ldots+\alpha_d=k} \|D^\alpha\varphi\|^2\right\}^{\frac{1}{2}}$ and norm
  $\|\varphi\|_{k,\Omega} = \|\varphi\|_k = \left\{\sum_{\ell=0}^k |\varphi|_\ell^2\right\}^{\frac{1}{2}}$
- Norms of vector- or tensor-valued functions are defined component-wise.
- $H_0^1(\Omega) = \{\varphi \in H^1(\Omega) : \varphi = 0 \text{ on } \Gamma = \partial\Omega\}$
- $V = \{\mathbf{v} \in H_0^1(\Omega)^d : \operatorname{div} \mathbf{v} = 0\}$
- $L_0^2(\Omega) = \{\varphi \in L^2(\Omega) : \int_\Omega \varphi = 0\}$

Poincaré, Friedrichs and Trace Inequalities

- Poincaré inequality: $\|\varphi\| \le c_P \operatorname{diam}(\Omega)|\varphi|_1$ for all $\varphi \in H^1(\Omega) \cap L_0^2(\Omega)$
- $c_P = \frac{1}{\pi}$ if Ω is convex.
- Friedrichs inequality: $\|\varphi\| \le c_F \operatorname{diam}(\Omega)|\varphi|_1$ for all $\varphi \in H_0^1(\Omega)$
- Trace inequality: $\|\varphi\|_\Gamma \le \left\{c_{T,1}(\Omega)\|\varphi\|^2 + c_{T,2}(\Omega)|\varphi|_1^2\right\}^{\frac{1}{2}}$ for all $\varphi \in H^1(\Omega)$
- $c_{T,1}(\Omega) \approx \operatorname{diam}(\Omega)^{-1}$, $c_{T,2}(\Omega) \approx \operatorname{diam}(\Omega)$ if Ω is a simplex or parallelepiped

Finite Element Meshes $\mathcal{T}$

- $\Omega \cup \Gamma$ is the union of all elements in $\mathcal{T}$.
- Affine equivalence: Each $K \in \mathcal{T}$ is either a triangle or a parallelogram, if $d = 2$, or a tetrahedron or a parallelepiped, if $d = 3$.
- Admissibility: Any two elements in $\mathcal{T}$ are either disjoint or share a vertex or a complete edge or – if $d = 3$ – a complete face.
- Shape-regularity: For every element K, the ratio of its diameter h_K to the diameter ρ_K of the largest ball inscribed into K is bounded independently of K.
- Mesh-size: $h = h_\mathcal{T} = \max_{K \in \mathcal{T}} h_K$

Finite Element Spaces

- $$R_k(K) = \begin{cases} \operatorname{span}\{x_1^{\alpha_1} \cdot \ldots \cdot x_d^{\alpha_d} : \alpha_1 + \ldots + \alpha_d \le k\} \\ \text{if } K \text{ is a triangle or a tetrahedron} \\ \operatorname{span}\{x_1^{\alpha_1} \cdot \ldots \cdot x_d^{\alpha_d} : \max\{\alpha_1, \ldots, \alpha_d\} \le k\} \\ \text{if } K \text{ is a parallelogram or a parallelepiped} \end{cases}$$
- $S^{k,-1}(\mathcal{T}) = \{\varphi : \Omega \to \mathbb{R} : \varphi\big|_K \in R_k(K)\,\forall K \in \mathcal{T}\}$
- $S^{k,0}(\mathcal{T}) = S^{k,-1}(\mathcal{T}) \cap C(\overline{\Omega})$
- $S_0^{k,0}(\mathcal{T}) = S^{k,0}(\mathcal{T}) \cap H_0^1(\Omega)$
  $= \{\varphi \in S^{k,0}(\mathcal{T}) : \varphi = 0 \text{ on } \Gamma\}$

Approximation Properties

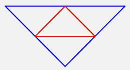
- $$\inf_{\varphi_\mathcal{T} \in S^{k,-1}(\mathcal{T})} \|\varphi - \varphi_\mathcal{T}\| \le c h^{k+1} |\varphi|_{k+1}$$
  $$\varphi \in H^{k+1}(\Omega),\ k \in \mathbb{N}$$
- $$\inf_{\varphi_\mathcal{T} \in S^{k,0}(\mathcal{T})} |\varphi - \varphi_\mathcal{T}|_j \le c h^{k+1-j} |\varphi|_{k+1}$$
  $$\varphi \in H^{k+1}(\Omega),\ j \in \{0,1\},\ k \in \mathbb{N}^*$$
- $$\inf_{\varphi_\mathcal{T} \in S_0^{k,0}(\mathcal{T})} |\varphi - \varphi_\mathcal{T}|_j \le c h^{k+1-j} |\varphi|_{k+1}$$
  $$\varphi \in H^{k+1}(\Omega) \cap H_0^1(\Omega),$$
  $$j \in \{0,1\},\ k \in \mathbb{N}^*$$

Vertices and Faces

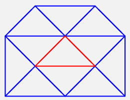
- $\mathcal{N}$: set of all element vertices
- $\mathcal{E}$: set of all $(d-1)$-dimensional element faces
- A subscript K, Ω or Γ to $\mathcal{N}$ or $\mathcal{E}$ indicates that only those vertices or faces are considered that are contained in the respective set.

Patches

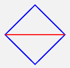
- $$\omega_K = \bigcup_{\mathcal{E}_K \cap \mathcal{E}_{K'} \neq \emptyset} K'$$



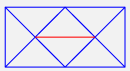
- $$\widetilde{\omega}_K = \bigcup_{\mathcal{N}_K \cap \mathcal{N}_{K'} \neq \emptyset} K'$$



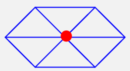
- $$\omega_E = \bigcup_{E \in \mathcal{E}_{K'}} K'$$



- $$\widetilde{\omega}_E = \bigcup_{\mathcal{N}_E \cap \mathcal{N}_{K'} \neq \emptyset} K'$$



- $$\omega_z = \bigcup_{z \in \mathcal{N}_{K'}} K'$$



Nodal Shape Functions

- λ_z denotes the nodal shape function associated with the vertex z.
- It is uniquely defined by the conditions
  $$\lambda_z \in S^{1,0}(\mathcal{T}),\ \lambda_z(z) = 1,\ \lambda_z(y) = 0\,\forall y \in \mathcal{N} \setminus \{z\}.$$
- ω_z is the support of λ_z.

A Quasi-Interpolation Operator

- Define the quasi-interpolation operator $R_{\mathcal{T}} : L^1(\Omega) \to S_0^{1,0}(\mathcal{T})$ by

$$R_{\mathcal{T}}\varphi = \sum_{z\in\mathcal{N}_\Omega} \lambda_z \overline{\varphi}_z \quad \text{with} \quad \overline{\varphi}_z = \frac{\int_{\omega_z} \varphi dx}{\int_{\omega_z} dx}.$$

- It has the following local approximation properties for all $\varphi \in H_0^1(\Omega)$

$$\|\varphi - R_{\mathcal{T}}\varphi\|_K \le c_{A1} h_K |\varphi|_{1,\widetilde{\omega}_K}$$
$$\|\varphi - R_{\mathcal{T}}\varphi\|_{\partial K} \le c_{A2} h_K^{\frac{1}{2}} |\varphi|_{1,\widetilde{\omega}_K}.$$

Proof of the Local Approximation Properties

- The Poincaré inequality implies for every vertex z $\|\varphi - \overline{\varphi}_z\|_{\omega_z} \le c_z \operatorname{diam}(\omega_z)|\varphi|_{1,\omega_z}$.
- The trace inequality yields for all faces E of all elements K $\|\varphi\|_E \le c_1 h_K^{-\frac{1}{2}}\|\varphi\|_K + c_2 h_K^{\frac{1}{2}}|\varphi|_{1,K}$.
- The properties of the nodal shape functions imply $\|\varphi - R_{\mathcal{T}}\varphi\|_K \le \sum_{z\in\mathcal{N}_K} \|\varphi - \overline{\varphi}_z\|_K + \sum_{z\in\mathcal{N}_{K,\Gamma}} \|\overline{\varphi}_z\|_K$.
- The first term is bounded using the Poincaré inequality, the second one using $\varphi \in H_0^1(\Omega)$.

Bubble Functions

- Define element and face bubble functions by

$$\psi_K = \alpha_K \prod_{z\in\mathcal{N}_K} \lambda_z, \quad \psi_E = \alpha_E \prod_{z\in\mathcal{N}_E} \lambda_z.$$

- The weights α_K and α_E are determined by the conditions

$$\max_{x\in K} \psi_K(x) = 1, \quad \max_{x\in E} \psi_E(x) = 1.$$

- K is the support of ψ_K; ω_E is the support of ψ_E.

Inverse Estimates for the Bubble Functions

For all elements K, all faces E and all polynomials φ the following inverse estimates are valid

$$c_{I1,k}\|\varphi\|_K \le \|\psi_K^{\frac{1}{2}}\varphi\|_K,$$
$$\|\nabla(\psi_K\varphi)\|_K \le c_{I2,k} h_K^{-1}\|\varphi\|_K,$$
$$c_{I3,k}\|\varphi\|_E \le \|\psi_E^{\frac{1}{2}}\varphi\|_E,$$
$$\|\nabla(\psi_E\varphi)\|_{\omega_E} \le c_{I4,k} h_E^{-\frac{1}{2}}\|\varphi\|_E,$$
$$\|\psi_E\varphi\|_{\omega_E} \le c_{I5,k} h_E^{\frac{1}{2}}\|\varphi\|_E.$$

Proof of the Inverse Estimates

- Transform the left hand-sides to the reference simplex or cube.
- Take into account that the left-hand sides define semi-norms.
- Invoke the equivalence of norms on finite dimensional spaces to prove the corresponding estimates on the reference element.
- Transform the right-hand sides back to the current element or face.

Jumps

- $\mathbf{n}_E$: a unit vector perpendicular to a given face E
- $[\varphi]_E$: jump of a given piece-wise continuous function across a given face E in the direction of $\mathbf{n}_E$
- $[\varphi]_E$ depends on the orientation of $\mathbf{n}_E$ but quantities of the form $[\mathbf{n}_E \cdot \nabla\varphi]_E$ are independent thereof.

Variational Formulation of the Stokes Equations

- A First Attempt
- Abstract Saddle-Point Problems
- Saddle-Point Formulation of the Stokes Equations

A Variational Formulation of the Stokes Equations

- Stokes equations with no-slip boundary condition

$$-\Delta\mathbf{u} + \operatorname{grad} p = \mathbf{f} \text{ in } \Omega, \ \operatorname{div}\mathbf{u} = 0 \text{ in } \Omega, \ \mathbf{u} = 0 \text{ on } \Gamma$$

- Multiply momentum equation with $\mathbf{v} \in V = \{\mathbf{w} \in H_0^1(\Omega)^d : \operatorname{div}\mathbf{w} = 0\}$, integrate over Ω and use integration by parts.
- Resulting variational formulation:
  Find $\mathbf{u} \in V$ such that for all $\mathbf{v} \in V$

$$\int_\Omega \nabla\mathbf{u} : \nabla\mathbf{v} = \int_\Omega \mathbf{f} \cdot \mathbf{v}$$

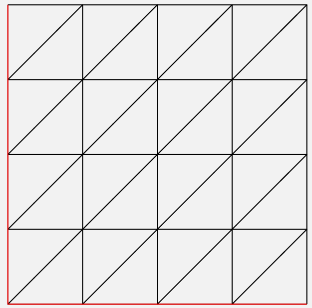
Corresponding Discretization

- Find $\mathbf{u}_{\mathcal{T}} \in V(\mathcal{T}) \subset V$ such that for all $\mathbf{v}_{\mathcal{T}} \in V(\mathcal{T})$

$$\int_\Omega \nabla \mathbf{u}_{\mathcal{T}} : \nabla \mathbf{v}_{\mathcal{T}} = \int_\Omega \mathbf{f} \cdot \mathbf{v}_{\mathcal{T}}$$

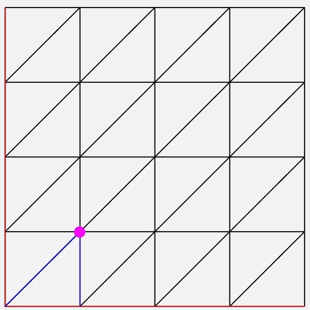
- Advantage:
  The discrete problem is symmetric positive definite.
- Disadvantage:
  The discrete problem gives no information on the pressure.
- Candidate for lowest order discretization:
  $V(\mathcal{T}) = S_0^{1,0}(\mathcal{T})^d \cap V$.

The Space $V(\mathcal{T}) = S_0^{1,0}(\mathcal{T})^d \cap V$



- $\Omega = (0,1)^2$
- $\mathcal{T}$ Courant triangulation consisting of $2N^2$ isosceles right-angled triangles with short sides of length $h = N^{-1}$
- $\mathbf{v}_{\mathcal{T}} \in S_0^{1,0}(\mathcal{T})^d \cap V$ arbitrary

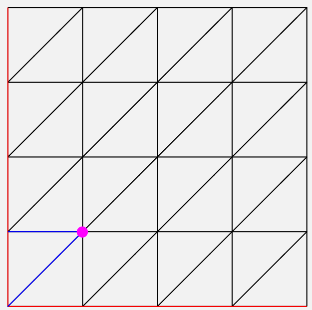
The Space $V(\mathcal{T}) = S_0^{1,0}(\mathcal{T})^d \cap V$



- $0 = \operatorname{div} \mathbf{v}_{\mathcal{T}}$ on K
- $0 = \int_K \operatorname{div} \mathbf{v}_{\mathcal{T}} = \int_{\partial K} \mathbf{n} \cdot \mathbf{v}_{\mathcal{T}}$
- $0 = \int_{\partial K} \mathbf{n} \cdot \mathbf{v}_{\mathcal{T}}$

$$= \sqrt{2} h \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \cdot \mathbf{v}_{\mathcal{T}}(x)$$
$$+ h \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \mathbf{v}_{\mathcal{T}}(x)$$
$$= h \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \mathbf{v}_{\mathcal{T}}(x)$$

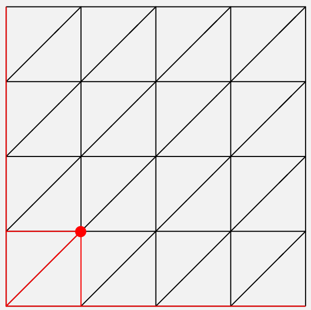
The Space $V(\mathcal{T}) = S_0^{1,0}(\mathcal{T})^d \cap V$



- $0 = \operatorname{div} \mathbf{v}_{\mathcal{T}}$ on K
- $0 = \int_K \operatorname{div} \mathbf{v}_{\mathcal{T}} = \int_{\partial K} \mathbf{n} \cdot \mathbf{v}_{\mathcal{T}}$
- $0 = \int_{\partial K} \mathbf{n} \cdot \mathbf{v}_{\mathcal{T}}$

$$= \sqrt{2} h \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cdot \mathbf{v}_{\mathcal{T}}(x)$$
$$+ h \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \mathbf{v}_{\mathcal{T}}(x)$$
$$= h \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \mathbf{v}_{\mathcal{T}}(x)$$

The Space $V(\mathcal{T}) = S_0^{1,0}(\mathcal{T})^d \cap V$



- $\mathbf{v}_\mathcal{T} = 0$ in bottom left square
- Sweeping through squares yields:

  $\mathbf{v}_\mathcal{T} = 0$ in Ω
- Hence:

  $S_0^{1,0}(\mathcal{T})^d \cap V = \{0\}$

Finite Element Subspaces of V

- In order to obtain a non-trivial space $S_0^{k,0}(\mathcal{T})^d \cap V$, the polynomial degree k must be at least 5.
- Despite the high polynomial degree, the approximation properties of $S_0^{k,0}(\mathcal{T})^d \cap V$ are rather poor.

Another Variational Formulation of the Stokes Equations

- Multiply the momentum equation with $\mathbf{v} \in H_0^1(\Omega)^d$, integrate over Ω and use integration by parts.
- Multiply the continuity equation with $q \in L_0^2(\Omega)$ and integrate over Ω.
- Resulting variational formulation:

  Find $\mathbf{u} \in H_0^1(\Omega)^d$ and $p \in L_0^2(\Omega)$ such that for all $\mathbf{v} \in H_0^1(\Omega)^d$ and $q \in L_0^2(\Omega)$

$$\int_\Omega \nabla \mathbf{u} : \nabla \mathbf{v} - \int_\Omega p \operatorname{div} \mathbf{v} = \int_\Omega \mathbf{f} \cdot \mathbf{v}$$
$$\int_\Omega q \operatorname{div} \mathbf{u} = 0$$

Corresponding Discretization

- Choose finite element spaces $X(\mathcal{T}) \subset H_0^1(\Omega)^d$ and $Y(\mathcal{T}) \subset L_0^2(\Omega)$.
- Find $\mathbf{u}_\mathcal{T} \in X(\mathcal{T})$ and $p_\mathcal{T} \in Y(\mathcal{T})$ such that for all $\mathbf{v}_\mathcal{T} \in X(\mathcal{T})$ and $q_\mathcal{T} \in Y(\mathcal{T})$

$$\int_\Omega \nabla \mathbf{u}_\mathcal{T} : \nabla \mathbf{v}_\mathcal{T} - \int_\Omega p_\mathcal{T} \operatorname{div} \mathbf{v}_\mathcal{T} = \int_\Omega \mathbf{f} \cdot \mathbf{v}$$
$$\int_\Omega q_\mathcal{T} \operatorname{div} \mathbf{u}_\mathcal{T} = 0$$

Questions

- Does the variational problem admit a unique solution?
- Does the discrete problem admit a unique solution?
- What is the quality of the approximation?
- What are good choices for the discrete spaces?

The Setting

- X and Y are Hilbert spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$.
- $a : X \times X \to \mathbb{R}$ and $b : X \times Y \to \mathbb{R}$ are continuous bilinear forms.
- $\ell : X \to \mathbb{R}$ and $\chi : Y \to \mathbb{R}$ are continuous linear functionals.
- Problem:

  Find $u \in X$ and $\lambda \in Y$ such that for all $v \in X$ and $\mu \in Y$

$$\begin{aligned} a(u,v) + b(v,\lambda) &= \langle \ell, v\rangle \\ b(u,\mu) &= \langle \chi, \mu\rangle \end{aligned} \qquad (S)$$

Auxiliary Operators and Spaces

- Define continuous linear operators $A : X \to X'$, $B : X \to Y'$, $B' : Y \to X'$ by setting for all $u, v \in X$, $\lambda \in Y$

$$\langle Au, v\rangle = a(u,v),\ \langle Bu, \lambda\rangle = b(u,\lambda),\ \langle B'\lambda, v\rangle = b(u,\lambda).$$

- Set

$$\begin{aligned} V &= \ker B, \\ V^\circ &= \{g \in X' : \langle g, v\rangle = 0\, \forall v \in V\}, \\ V^\perp &= \{u \in X : (u,v)_X = 0\, \forall v \in V\}. \end{aligned}$$

- Define the continuous linear operator $\pi : X' \to V'$ by

$$\langle \pi f, v\rangle = \langle f, v\rangle\ \forall v \in V.$$

The Inf-Sup Condition

The following conditions are equivalent:

1. There is a constant $\beta > 0$ such that (inf-sup condition)

$$\inf_{\lambda \in Y\setminus\{0\}} \sup_{u \in X\setminus\{0\}} \frac{b(u,\lambda)}{\|u\|_X \|\lambda\|_Y} \geq \beta.$$

2. B' is an isomorphism of Y onto V° and $\|B'\lambda\|_{X'} \geq \beta\|\lambda\|_Y$ for all $\lambda \in Y$.
3. B is an isomorphism of $V^\perp$ onto Y' and $\|Bu\|_{Y'} \geq \beta\|u\|_X$ for all $u \in X$.

Motivation of the Inf-Sup Condition

Assume that $X = \mathbb{R}^n$, $Y = \mathbb{R}^m$ with $m < n$ and $b(u, \lambda) = \lambda^T Bu$ with a rectangular matrix $B \in \mathbb{R}^{m\times n}$. Then the following conditions are equivalent:

- B has maximal rang m.
- The rows of B are linearly independent.
- $\lambda^T Bu = 0$ for all $u \in \mathbb{R}^n$ implies $\lambda = 0$.
- $\inf_\lambda \sup_u \frac{\lambda^T Bu}{|u||\lambda|} > 0$.
- The linear system $B^T\lambda = 0$ only admits the trivial solution.
- For every $f \in \mathbb{R}^m$ there is a unique $u \in \mathbb{R}^n$ which is orthogonal to $\ker B$ and which satisfies $Bu = f$.

Proof of the Equivalences

1. $\Rightarrow$ 2. Condition (1), the definition of B' and $\|\cdot\|_{X'}$ imply

$$\|B'\lambda\|_{X'} = \sup_{u\in X\setminus\{0\}} \frac{b(u,\lambda)}{\|u\|_X} \geq \beta\|\lambda\|_Y.$$

Hence, B' is injective and its range is closed. The closed graph theorem then proves (2).

2. $\Rightarrow$ 1. This is a consequence of the above equality.

2. $\Leftrightarrow$ 3. From the definitions of V° and $V^\perp$ one concludes that V° and $(V^\perp)'$ are isometric. Hence, B is an isomorphism of $V^\perp$ onto Y' if and only if B' is an isomorphism of $(Y')' \simeq Y$ onto $(V^\perp)' \simeq V^\circ$ and both isomorphisms have the same norm.

Well-Posedness of Problem (S)

- Problem (S) admits a unique solution for every right-hand side if and only if
  - (i) πA is an isomorphism of V onto V' and
  - (ii) b satisfies the inf-sup condition.
- If problem (S) is well-posed, its solution satisfies

$$\|u\|_X + \|\lambda\|_Y \leq c\big\{\|\ell\|_{X'} + \|\chi\|_{Y'}\big\}.$$

The constant c grows with β^{-1}.

Proof of " $\Leftarrow$ "

- Due to (ii) there is a unique $u_0 \in V^\perp$ with $Bu_0 = \chi$ and $\|u_0\|_X \leq \frac{1}{\beta}\|\chi\|_{Y'}$.
- Due to (i) there is a unique $w \in V$ with $\pi Aw = \pi(\ell - Au_0)$ and $\|w\|_X \leq \|(\pi A)^{-1}\|_{\mathcal{L}(V',V)}\|\ell - Au_0\|_{X'}$.
- $u = u_0 + w$ satisfies $\pi(\ell - Au) = 0$ whence $\ell - Au \in V^\circ$.
- Due to (ii) there is a unique $\lambda \in Y$ with $B'\lambda = \ell - Au$ and $\|\lambda\|_Y \leq \frac{1}{\beta}\|\ell - Au\|_{X'}$.
- u, λ solve (S) and satisfy the stability estimate.

Proof of " $\Rightarrow$ "

- Problem (S) with $\ell = 0$ and arbitrary $\chi \in Y'$ admits a unique solution. Hence $Y' = \text{range}\, B$. The open mapping theorem proves that B is an isomorphism and thus establishes (ii).
- Consider a $u \in V$ with $\pi A u = 0$. Due to (ii), there is a unique $\lambda \in Y$ with $B'\lambda = -Au$. Thus u, λ solve (S) with homogeneous right-hand side. Hence, $u = 0$ and πA is injective.
- Due to the Hahn-Banach theorem, for every $g \in V'$, there is an $\ell \in X'$ with $\pi\ell = g$. Problem (S) admits a unique solution u, λ for the right-hand side ℓ, $\chi = 0$. Hence, there is a $u \in X$ with $\pi A u = g$ and πA is surjective.
- The open mapping theorem proves that πA is an isomorphism and thus establishes (i).

Coercive Forms a

Assume that a is symmetric and coercive on X, i.e. there is an $\alpha > 0$ such that $a(u,u) \geq \alpha \|u\|_X^2$ holds for all $u \in X$. Then:

- πA is an isomorphism and $\|(\pi A)^{-1}\|_{\mathcal{L}(V',V)} \leq \frac{1}{\alpha}$.
- Problem (S) is well-posed if and only if the form b satisfies the inf-sup condition.
- The solution of problem (S) is the unique saddle-point of the functional $\mathcal{L}(u,\lambda) = \frac{1}{2}a(u,u) + b(u,\lambda) - \langle \ell, u\rangle - \langle \chi, \lambda\rangle$.
- The solution u of (S) minimizes the functional $J(u) = \frac{1}{2}a(u,u) - \langle \ell, u\rangle$ under the constraint $b(u,\mu) = \langle \chi, \mu\rangle$ for all $\mu \in Y$.

Discretization of Saddle-Point Problems

- Replace X and Y by finite dimensional subspaces X_n and Y_n.
- Resulting discrete problem:

  Find $u_n \in X_n$ and $\lambda_n \in Y_n$ such that for all $v_n \in X_n$ and $\mu_n \in Y_n$

$$\begin{aligned} a(u_n, v_n) + b(v_n, \lambda_n) &= \langle \ell, v_n\rangle \\ b(u_n, \mu_n) &= \langle \chi, \mu_n\rangle \end{aligned} \qquad (S_n)$$

Well-Posedness of Problem (S_n)

Assume for simplicity that the form a is coercive on X. Then:

- Problem (S_n) is well posed if and only if the form b satisfies the discrete inf-sup condition

$$\inf_{\lambda_n \in Y_n \setminus \{0\}} \sup_{u_n \in X_n \setminus \{0\}} \frac{b(u_n, \lambda_n)}{\|u_n\|_X \|\lambda_n\|_Y} \geq \beta_n > 0.$$

- If problem (S_n) is well-posed, its solution satisfies

$$\|u_n\|_X + \|\lambda_n\|_Y \leq c\big\{\|\ell\|_{X'} + \|\chi\|_{Y'}\big\}.$$

  The constant c grows with β_n^{-1}.

Error Estimates

- Assume that a is coercive and that b satisfies both the inf-sup condition and the discrete inf-sup condition.
- Denote by u, λ the unique solution of problem (S) and by u_n, λ_n the unique solution of problem (S_n).
- Then there is a constant c which grows with β_n^{-1} such that

$$\|u-u_n\|_X + \|\lambda-\lambda_n\|_Y \le c\Big\{\inf_{v_n\in X_n}\|u-v_n\|_X + \inf_{\mu_n\in Y_n}\|\lambda-\mu_n\|_Y\Big\}$$

Proof of the Error Estimates

- For any $v_n \in X_n$, $\mu_n \in Y_n$ define $\widetilde{\ell} \in X'$, $\widetilde{\chi} \in Y'$ by

$$\langle\widetilde{\ell}, v\rangle = a(u-v_n, v) + b(v, \lambda-\mu_n),$$
$$\langle\widetilde{\chi}, \mu\rangle = b(u-v_n, \mu).$$

- Then $\|\widetilde{\ell}\|_{X'} + \|\widetilde{\chi}\|_{Y'} \le c\big\{\|u-v_n\|_X + \|\lambda-\mu_n\|_Y\big\}$.
- Subtracting problems (S) and (S_n) gives for every $w_n \in X_n$ and $\rho_n \in Y_n$

$$\langle\widetilde{\ell}, w_n\rangle = a(u_n-v_n, w_n) + b(w_n, \lambda_n-\mu_n),$$
$$\langle\widetilde{\chi}, \rho_n\rangle = b(u_n-v_n, \rho_n).$$

- The stability estimate for problem (S_n) and the triangle inequality now prove the error estimate.

A Duality Argument

- H is another Hilbert space with norm $\|\cdot\|_H$ such that X is dense in H with continuous injection.
- For every $g \in H$ denote by u_g, λ_g the solution of problem (S) with $\ell = g$ and $\chi = 0$.
- Then u and u_n satisfy the error estimate

$$\|u-u_n\|_H \le c\big\{\|u-u_n\|_X + \|\lambda-\lambda_n\|_Y\big\}\cdot \sup_{g\in H\setminus\{0\}}\frac{1}{\|g\|_H}\Big\{\inf_{v_n\in X_n}\|u-v_n\|_X + \inf_{\mu_n\in Y_n}\|\lambda-\mu_n\|_Y\Big\}.$$

Proof

- The density of X in H implies

$$\|u-u_n\|_H = \sup_{g\in H\setminus\{0\}}\frac{(g, u-u_n)_H}{\|g\|_H}.$$

- Subtracting (S) and (S_n) and using the definition of u_g, λ_g yields for every $v_n \in X_n$, $\mu_n \in Y_n$

$$\begin{aligned}(g, u-u_n)_H &= a(u-u_n, u_g) + b(u-u_n, \lambda_g) + \underbrace{b(u_g, \lambda-\lambda_n)}_{=0}\\ &= a(u-u_n, u_g-v_n) + b(u_g-v_n, \lambda-\lambda_n) + b(u-u_n, \lambda_g-\mu_n)\\ &\quad + \underbrace{a(u-u_n, v_n) + b(v_n, \lambda-\lambda_n)}_{=0} + \underbrace{b(u-u_n, \mu_n)}_{=0}.\end{aligned}$$

Saddle-Point Formulation of the Stokes Equations

- The saddle-point formulation of the Stokes equations fits into the abstract framework with:
  - $X = H_0^1(\Omega)^d$, $Y = L_0^2(\Omega)$, $H = L^2(\Omega)^d$
  - $a(\mathbf{u},\mathbf{v}) = \int_\Omega \nabla\mathbf{u} : \nabla\mathbf{v}$, $b(\mathbf{u},p) = -\int_\Omega p \operatorname{div}\mathbf{u}$
  - $\langle \ell, \mathbf{v}\rangle = \int_\Omega \mathbf{f}\cdot\mathbf{v}$, $\chi = 0$
- The bilinear form a is coercive on X. Hence, we only have to ascertain the inf-sup condition

$$\inf_{p\in L_0^2(\Omega)\setminus\{0\}} \sup_{\mathbf{u}\in H_0^1(\Omega)^d\setminus\{0\}} \frac{\int_\Omega p \operatorname{div}\mathbf{u}}{|\mathbf{u}|_1\|p\|} \geq \beta > 0.$$

A Proof in $\mathbb{R}^2$

- Assume that $\Omega \subset \mathbb{R}^2$ is either convex or has a C^2 boundary.
- Choose an arbitrary $p \in L_0^2(\Omega)$.
- Set $\mathbf{v} = \nabla\varphi$ where $\varphi \in H^2(\Omega) \cap L_0^2(\Omega)$ is the unique weak solution of the Neumann problem

$$\Delta\varphi = p \quad \text{in } \Omega, \qquad \frac{\partial\varphi}{\partial n} = 0 \quad \text{on } \Gamma.$$

- Set $\mathbf{w} = (\frac{\partial\psi}{\partial x_2}, -\frac{\partial\psi}{\partial x_1})$ where $\psi \in H^2(\Omega)$ is the unique weak solution of the biharmonic equation

$$\Delta^2\psi = 0 \quad \text{in } \Omega, \qquad \psi = 0 \quad \text{on } \Gamma, \qquad \frac{\partial\psi}{\partial n} = \mathbf{v}\cdot\mathbf{t} \quad \text{on } \Gamma.$$

- Set $\mathbf{u} = \mathbf{v} + \mathbf{w}$. Then $\operatorname{div}\mathbf{u} = p$ and $|\mathbf{u}|_1 \leq c\|p\|$.

A Proof by Duvaut, Lions and Nečas. 1^{st} Step

$\|p\| \leq c(\Omega)\{\|p\|_{-1} + \|\nabla p\|_{-1}\}$

- Set $X(\Omega) = \{p \in H^{-1}(\Omega) : \nabla p \in H^{-1}(\Omega)^d\}$ equipped with $|||p||| = \|p\|_{-1} + \|\nabla p\|_{-1}$.
- The definition $H^{-1}(\Omega)$ and the open mapping theorem imply that it suffices to prove the inclusion $X(\Omega) \subset L^2(\Omega)$.
- Due to the characterization of Sobolev spaces by Fourier transforms, the inclusion holds for $\mathbb{R}^d$.
- Using suitable reflections shows that the inclusion also holds for C^∞-functions on $\mathbb{R}^{d-1}\times\mathbb{R}_+$.
- The Hahn-Banach theorem implies that $C^\infty(\mathbb{R}^{d-1}\times\mathbb{R}_+)$ is dense in $X(\mathbb{R}^{d-1}\times\mathbb{R}_+)$.
- Combining the previous results with suitable partitions of unity establishes the inclusion for all Lipschitz domains Ω.

A Proof by Duvaut, Lions and Nečas. 2^{nd} Step

$\|p\| \leq c(\Omega)\|\nabla p\|_{-1}$

- Assume the contrary.
- Then there is a sequence (p_n) in $L_0^2(\Omega)$ with $\|p_n\| = 1$ and $\|\nabla p_n\|_{-1} \leq \frac{1}{n}$ for all n.
- Since $H_0^1(\Omega)$ is compactly embedded in $L^2(\Omega)$, the latter space is compactly embedded in $H^{-1}(\Omega)$.
- Hence, there is subsequence (p_{n_k}) such that $p_{n_k} \to p$ strongly in H^{-1}, $p_{n_k} \to p$ weakly in L^2 and $\int_\Omega \nabla p_{n_k}\cdot\mathbf{v} \to \int_\Omega \nabla p\cdot\mathbf{v}$ for all C^∞ vector-fields $\mathbf{v}$.
- This proves $\nabla p = 0$ and, since $p_n \in L_0^2(\Omega)$, $p = 0$.
- This contradicts the estimate on the previous slide.

A Proof by Duvaut, Lions and Nečas. 3rd Step

The inf-sup condition is fulfilled.

- The operator grad : $L_0^2(\Omega) \to H^{-1}(\Omega)^d$ is injective and continuous.
- The previous result implies that range(grad) is a closed subspace of $H^{-1}(\Omega)^d$.
- The open mapping theorem implies that grad is an isomorphism of $L_0^2(\Omega)$ onto range(grad).
- The closed range theorem implies that range(grad) $= \ker(\mathrm{div})^\circ = V^\circ$.
- Due to the abstract results, this proves the inf-sup condition.

A Proof by Bogovskii

Assume that the domain Ω is the union of a finite number of (eventually overlapping) subdomains which are star-shaped with respect to an inscribed ball. Then the inf-sup condition holds.

- A suitable additive decomposition of the pressure and velocity shows that it suffices to establish the inf-sup condition for a single subdomain.
- Consider a subdomain ω which is star-shaped with respect to an open ball K with $\overline{K} \subset \omega$. Choose a C^∞-function φ with support in K and $\int_K \varphi = 1$.
- Properties of singular integrals imply that

$$\mathbf{u}(x) = \int_\omega p(y) \frac{x-y}{|x-y|^d} \int_{|x-y|}^\infty \varphi\big(y + t\frac{x-y}{|x-y|}\big) t^{d-1} dt dy$$

satisfies $\mathrm{div}\, \mathbf{u} = p$ in ω and $|\mathbf{u}|_{1,\omega} \le c\, \mathrm{diam}(\omega) \|p\|_\omega$.

A Regularity Result

- Assume that the boundary Γ is of class C^{m+2} and that $\mathbf{f} \in H^m(\Omega)^d$. Then the weak solution of the Stokes problem satisfies:
  - $\mathbf{u} \in H^{m+2}(\Omega)^d \cap H_0^1(\Omega)^d$, $p \in H^{m+1}(\Omega) \cap L_0^2(\Omega)$,
  - $\|\mathbf{u}\|_{m+2} + \|p\|_{m+1} \le c(\Omega) \|\mathbf{f}\|_m$.
- If Ω is a convex polyhedron, the above regularity result holds with $m = 0$.

Finite Element Discretization

- The finite element discretization of the Stokes equations fits into the abstract framework with
  $X_n = X(\mathcal{T})$, $Y_n = Y(\mathcal{T})$
- The bilinear form a is coercive on X. Hence, we only have to ascertain the discrete inf-sup condition

$$\inf_{p_\mathcal{T} \in Y(\mathcal{T}) \setminus \{0\}} \sup_{\mathbf{u}_\mathcal{T} \in X(\mathcal{T}) \setminus \{0\}} \frac{\int_\Omega p_\mathcal{T}\, \mathrm{div}\, \mathbf{u}_\mathcal{T}}{|\mathbf{u}_\mathcal{T}|_1 \|p_\mathcal{T}\|} \ge \beta_\mathcal{T} > 0.$$

- In order to obtain optimal error estimates, the discretization must be uniformly stable, i.e. $\beta_\mathcal{T} \ge \widetilde{\beta} > 0$ for all $\mathcal{T}$.

Resulting Error Estimates

- Assume:
  - $\mathbf{u} \in H^{k+1}(\Omega)^d \cap H_0^1(\Omega)^d$, $p \in H^k(\Omega) \cap L_0^2(\Omega)$.
  - The discretization is uniformly stable.
  - $S^{k,0}(\mathcal{T})^d \subset X(\mathcal{T})$.
  - $S^{k-1,0}(\mathcal{T}) \cap L_0^2(\Omega) \subset Y(\mathcal{T})$ or $S^{k-1,-1}(\mathcal{T}) \cap L_0^2(\Omega) \subset Y(\mathcal{T})$.
- Then:

$$|\mathbf{u} - \mathbf{u}_{\mathcal{T}}|_1 + \|p - p_{\mathcal{T}}\| \le ch^k \Big\{ |\mathbf{u}|_{k+1} + |p|_k \Big\}.$$

- If in addition Ω is a convex polyhedron, then:

$$\|\mathbf{u} - \mathbf{u}_{\mathcal{T}}\| \le ch^{k+1} \Big\{ |\mathbf{u}|_{k+1} + |p|_k \Big\}.$$

Approximation of the Space V

- The space $V = \{\mathbf{v} \in H_0^1(\Omega)^d : \operatorname{div} \mathbf{v} = 0\}$ is approximated by

$$V(\mathcal{T}) = \Big\{ \mathbf{v}_{\mathcal{T}} \in X(\mathcal{T}) : \int_\Omega p_{\mathcal{T}} \operatorname{div} \mathbf{v}_{\mathcal{T}} = 0 \,\forall p_{\mathcal{T}} \in Y(\mathcal{T}) \Big\}.$$

- For almost all discretizations used in practice $V(\mathcal{T})$ is not contained in V.
- In this sense, all these discretizations are non-conforming and not fully conservative.

Discretization of the Stokes Equations

- A Second Attempt
- Stable Finite Element Pairs
- Petrov-Galerkin Methods
- Non-Conforming Discretizations
- Stream-Function Formulation

The $P1/P0$-Element

- $\mathcal{T}$ is a triangulation of a two-dimensional domain Ω.
- $X(\mathcal{T}) = S_0^{1,0}(\mathcal{T})^d$, $Y(\mathcal{T}) = S^{0,-1}(\mathcal{T}) \cap L_0^2(\Omega)$
- Every solution $\mathbf{u}_{\mathcal{T}} \in X(\mathcal{T})$, $p_{\mathcal{T}} \in Y(\mathcal{T})$ of every discrete Stokes problem satisfies:
  - $\operatorname{div} \mathbf{u}_{\mathcal{T}}$ is element-wise constant and $\int_K \operatorname{div} \mathbf{u}_{\mathcal{T}} = 0$ for every $K \in \mathcal{T}$.
  - Hence, $\operatorname{div} \mathbf{u}_{\mathcal{T}} = 0$.
  - Our first attempt yields $\mathbf{u}_{\mathcal{T}} = 0$.
- Hence, this pair of finite element spaces is not stable and not suited for the discretization of the Stokes problem.

The $Q1/Q0$-Element

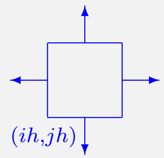
- $\mathcal{T}$ is a partition of the unit square $\Omega = (0,1)^2$ into N^2 squares with sides of length $h = N^{-1}$ where $N \geq 2$ is even.
- $X(\mathcal{T}) = S_0^{1,0}(\mathcal{T})^d$, $Y(\mathcal{T}) = S^{0,-1}(\mathcal{T}) \cap L_0^2(\Omega)$

-1	$+1$	-1	$+1$
$+1$	-1	$+1$	-1
-1	$+1$	-1	$+1$
$+1$	-1	$+1$	-1

- Denote by K_{ij} the square with bottom left corner (ih, jh).
- $\widehat{p}_{\mathcal{T}} \in Y(\mathcal{T})$ is the pressure with $p_{\mathcal{T}}|_{K_{ij}} = (-1)^{i+j}$ (checker-board mode)
- Then $\int_\Omega \widehat{p}_{\mathcal{T}} \operatorname{div} \mathbf{v}_{\mathcal{T}} = 0$ for every $\mathbf{v}_{\mathcal{T}} \in X(\mathcal{T})$ (checker-board instability).

- Hence, this pair of finite element spaces is not stable and not suited for the discretization of the Stokes problem.

Proof of the Checker-Board Instability



$$\int_{K_{ij}} \operatorname{div} \mathbf{v}_{\mathcal{T}} dx = \int_{\partial K_{ij}} \mathbf{v}_{\mathcal{T}} \cdot \mathbf{n}_{K_{ij}} dS$$

$$= \frac{h}{2}\Big\{ \mathbf{v}_{\mathcal{T}}(ih, jh) \cdot \begin{pmatrix} -1 \\ -1 \end{pmatrix} + \mathbf{v}_{\mathcal{T}}((i+1)h, jh) \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$+ \mathbf{v}_{\mathcal{T}}((i+1)h, (j+1)h) \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \mathbf{v}_{\mathcal{T}}(ih, (j+1)h) \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} \Big\}$$

$$\Rightarrow \int_\Omega \widehat{p}_{\mathcal{T}} \operatorname{div} \mathbf{v}_{\mathcal{T}} dx = \sum_{i,j} (-1)^{i+j} \int_{K_{ij}} \operatorname{div} \mathbf{v}_{\mathcal{T}} dx = 0$$

Conclusions

The velocity space must contain enough degrees of freedom in order to balance

- element-wise the gradient of the pressure,
- face-wise the jump of the pressure.

An Auxiliary Result

- Define a mesh-dependent norm on $S^{k,-1}(\mathcal{T})$ by

$$|\varphi|_{1,\mathcal{T}} = \Big\{ \sum_{K \in \mathcal{T}} h_K^2 \|\nabla \varphi\|_K^2 + \sum_{E \in \mathcal{E}} h_E \|[\varphi]_E\|_E^2 \Big\}^{\frac{1}{2}}.$$

- Assume that:
  - $S_0^{1,0}(\mathcal{T})^d \subset X(\mathcal{T})$,
  - $Y(\mathcal{T}) \subset S^{k,-1}(\mathcal{T})$ for some k,
  - There is a constant $\widetilde{\beta} > 0$ independent of $\mathcal{T}$ such that

$$\inf_{p_{\mathcal{T}} \in Y(\mathcal{T}) \setminus \{0\}} \sup_{\mathbf{u}_{\mathcal{T}} \in X(\mathcal{T}) \setminus \{0\}} \frac{\int_\Omega p_{\mathcal{T}} \operatorname{div} \mathbf{u}_{\mathcal{T}}}{|\mathbf{u}_{\mathcal{T}}|_1 |p_{\mathcal{T}}|_{1,\mathcal{T}}} \geq \widetilde{\beta}.$$

- Then the pair $X(\mathcal{T})$, $Y(\mathcal{T})$ is uniformly stable.

Proof of the Auxiliary Result. 1[st] Step

- Choose a pressure $p_\mathcal{T} \in Y(\mathcal{T})$ with $\|p_\mathcal{T}\| = 1$.
- Due to the well-posedness of the Stokes problem, there is a velocity $\mathbf{u} \in H_0^1(\mathcal{T})^d$ with

$$|\mathbf{u}|_1 = 1 \quad \text{and} \quad \int_\Omega p_\mathcal{T} \operatorname{div} \mathbf{u} \geq \beta.$$

- $R_\mathcal{T}\mathbf{u}$ satisfies

$$|R_\mathcal{T}\mathbf{u}|_1 \leq c_1 |\mathbf{u}|_1 = c_1,$$

$$\int_\Omega p_\mathcal{T} \operatorname{div}(R_\mathcal{T}\mathbf{u}) = \int_\Omega p_\mathcal{T} \operatorname{div} \mathbf{u} + \int_\Omega p_\mathcal{T} \operatorname{div}(R_\mathcal{T}\mathbf{u} - \mathbf{u}) \geq \beta + \int_\Omega p_\mathcal{T} \operatorname{div}(R_\mathcal{T}\mathbf{u} - \mathbf{u}).$$

Proof of the Auxiliary Result. 2[nd] Step

- Integration by parts and the properties of $R_\mathcal{T}$ imply

$$\int_\Omega p_\mathcal{T} \operatorname{div}(R_\mathcal{T}\mathbf{u} - \mathbf{u}) = \sum_{K\in\mathcal{T}} \int_K \nabla p_\mathcal{T} \cdot (\mathbf{u} - R_\mathcal{T}\mathbf{u}) + \sum_{E\in\mathcal{E}} \int_E [p_\mathcal{T}]_E (R_\mathcal{T}\mathbf{u} - \mathbf{u}) \cdot \mathbf{n}_E \leq c_2 |p_\mathcal{T}|_{1,\mathcal{T}} |\mathbf{u}|_1.$$

- The last two estimates yield

$$\sup_{\mathbf{u}_\mathcal{T} \in X(\mathcal{T})\setminus\{0\}} \frac{\int_\Omega p_\mathcal{T} \operatorname{div} \mathbf{u}_\mathcal{T}}{|\mathbf{u}_\mathcal{T}|_1} \geq \frac{1}{c_1}\{\beta - c_2 |p_\mathcal{T}|_{1,\mathcal{T}}\}.$$

Proof of the Auxiliary Result. 3[rd] Step

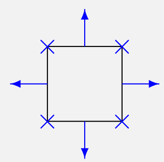
- The previous estimate and the third assumption imply

$$\sup_{\mathbf{u}_\mathcal{T} \in X(\mathcal{T})\setminus\{0\}} \frac{\int_\Omega p_\mathcal{T} \operatorname{div} \mathbf{u}_\mathcal{T}}{|\mathbf{u}_\mathcal{T}|_1} \geq \max\left\{\widetilde{\beta} |p_\mathcal{T}|_{1,\mathcal{T}}, \frac{1}{c_1}\{\beta - c_2 |p_\mathcal{T}|_{1,\mathcal{T}}\}\right\} \geq \min_{z\geq 0} \max\left\{\widetilde{\beta} z, \frac{1}{c_1}\{\beta - c_2 z\}\right\} = \frac{\beta\widetilde{\beta}}{c_1\widetilde{\beta} + c_2}.$$

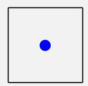
The Bernardi-Raugel Element

The following pair of finite element spaces is uniformly stable:

- $\mathcal{T}$ is any affine equivalent partition of a two or three dimensional domain.
- $X(\mathcal{T}) = S_0^{1,0}(\mathcal{T})^d \oplus \operatorname{span}\{\psi_E \mathbf{n}_E : E \in \mathcal{E}\}$



- $Y(\mathcal{T}) = S^{0,-1}(\mathcal{T}) \cap L_0^2(\Omega)$



Proof of the Stability of the Bernardi-Raugel Element

- For $p_\mathcal{T} \in Y(\mathcal{T})$ set $\mathbf{u}_\mathcal{T} = \sum_{E \in \mathcal{E}} h_E [p_\mathcal{T}]_E \mathbf{n}_E$.
- Integration by parts element-wise and the properties of the bubble-functions imply

$$\int_\Omega p_\mathcal{T} \operatorname{div} \mathbf{u}_\mathcal{T} = \sum_{E \in \mathcal{E}} \int_E [p_\mathcal{T}]_E \mathbf{u}_\mathcal{T} \cdot \mathbf{n}_E \geq \widetilde{\beta} |p_\mathcal{T}|^2_{1,\mathcal{T}}$$

and

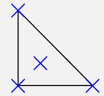
$$|\mathbf{u}_\mathcal{T}|_1 \leq c |p_\mathcal{T}|_{1,\mathcal{T}}.$$

- Hence, the auxiliary result proves the stability.

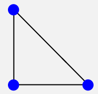
The Mini Element of Brezzi-Fortin

The following pair of finite element spaces is uniformly stable:

- $\mathcal{T}$ is any simplicial partition of a two or three dimensional domain.
- $X(\mathcal{T}) = S_0^{1,0}(\mathcal{T})^d \oplus \operatorname{span}\{\psi_K : K \in \mathcal{T}\}^d$



- $Y(\mathcal{T}) = S^{1,0}(\mathcal{T}) \cap L_0^2(\Omega)$



Proof of the Stability of the Mini Element

- For $p_\mathcal{T} \in Y(\mathcal{T})$ set $\mathbf{u}_\mathcal{T} = -\sum_{K \in \mathcal{T}} h_K^2 \psi_K \nabla p_\mathcal{T}$.
- Integration by parts element-wise and the properties of the bubble-functions imply

$$\int_\Omega p_\mathcal{T} \operatorname{div} \mathbf{u}_\mathcal{T} = -\sum_{K \in \mathcal{T}} \int_K \nabla p_\mathcal{T} \cdot \mathbf{u}_\mathcal{T} \geq \widetilde{\beta} |p_\mathcal{T}|^2_{1,\mathcal{T}}$$

and

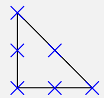
$$|\mathbf{u}_\mathcal{T}|_1 \leq c |p_\mathcal{T}|_{1,\mathcal{T}}.$$

- Hence, the auxiliary result proves the stability.

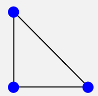
The Hood-Taylor Element

The following pair of finite element spaces is uniformly stable:

- $\mathcal{T}$ is any simplicial partition of a two or three dimensional domain.
- $X(\mathcal{T}) = S_0^{2,0}(\mathcal{T})^d$



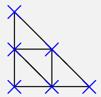
- $Y(\mathcal{T}) = S^{1,0}(\mathcal{T}) \cap L_0^2(\Omega)$



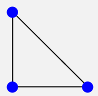
The Modified Hood-Taylor Element

The following pair of finite element spaces is uniformly stable:

- $\mathcal{T}$ is any simplicial partition of a two or three dimensional domain.
- $\mathcal{T}/2$ is obtained from $\mathcal{T}$ by uniform refinement connecting the midpoints of edges.
- $X(\mathcal{T}) = S_0^{1,0}(\mathcal{T}/2)^d$



- $Y(\mathcal{T}) = S^{1,0}(\mathcal{T}) \cap L_0^2(\Omega)$



Proof of the Stability of the Hood-Taylor and Modified Hood-Taylor Elements

- For every $p_\mathcal{T} \in Y(\mathcal{T})$ there is a $\mathbf{u}_\mathcal{T} \in X(\mathcal{T})$ such that $\mathbf{u}_\mathcal{T}$ coincides with the tangential derivative of $p_\mathcal{T}$ at the midpoints of edges.
- Bercovier and Pironneau proved in 1979 that with this choice of $\mathbf{u}_\mathcal{T}$ the third condition of the auxiliary result is fulfilled.
- Hence, the auxiliary result proves the stability.

A Catalogue of Stable Elements

The previous arguments can be modified to prove that the following pairs of spaces are uniformly stable on any affine equivalent partition in $\mathbb{R}^d$, $d \ge 2$:

- $X(\mathcal{T}) = S_0^{k,0}(\mathcal{T})^d \oplus \operatorname{span}\{\varphi\psi_E \mathbf{n}_E : E \in \mathcal{E},\ \varphi \in R_{k-1}(E)\}$
  $\oplus \operatorname{span}\{\rho\psi_K : K \in \mathcal{T},\ \rho \in R_{k-2}(K)\}^d$,
  $Y(\mathcal{T}) = S^{k-1,-1}(\mathcal{T}) \cap L_0^2(\Omega)$, $k \ge 2$
- $X(\mathcal{T}) = S_0^{k+d-1,0}(\mathcal{T})^d$, $Y(\mathcal{T}) = S^{k-1,-1}(\mathcal{T}) \cap L_0^2(\Omega)$, $k \ge 2$
- $X(\mathcal{T}) = S_0^{k,0}(\mathcal{T})^d$, $Y(\mathcal{T}) = S^{k-1,0}(\mathcal{T}) \cap L_0^2(\Omega)$, $k \ge 3$

Properties of the Mini Element

- $\displaystyle\int_\Omega \nabla\psi_K \cdot \nabla\psi_{K'} = 0$ for all $K \ne K'$
- $\displaystyle\int_\Omega \nabla\varphi \cdot \nabla\psi_K = \int_K \nabla\varphi \cdot \nabla\psi_K = -\int_K \Delta\varphi\psi_K = 0$
  for all $\varphi \in S^{1,0}(\mathcal{T})$, $K \in \mathcal{T}$
- Hence, the bubble part of the velocity of the mini element can be eliminated by static condensation.
- The resulting system only incorporates linear velocities and pressures.

The Mini Element with Static Condensation

- Original system:

$$\begin{pmatrix} A_\ell & 0 & B_\ell^T \\ 0 & D_b & B_b^T \\ B_\ell & B_b & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_\ell \\ \mathbf{u}_b \\ p \end{pmatrix} = \begin{pmatrix} \mathbf{f}_\ell \\ \mathbf{f}_b \\ 0 \end{pmatrix}$$

- System with static condensation:

$$\begin{pmatrix} A_\ell & B_\ell^T \\ B_\ell & -B_b D_b^{-1} B_b^T \end{pmatrix} \begin{pmatrix} \mathbf{u}_\ell \\ p \end{pmatrix} = \begin{pmatrix} \mathbf{f}_\ell \\ -B_b D_b^{-1} \mathbf{f}_b \end{pmatrix}$$

- A straightforward calculation yields:

$$\left(B_b D_b^{-1} B_b^T\right)_{i,j} \approx \sum_{K\in\mathcal{T}} h_K^2 \int_K \nabla\lambda_i \cdot \nabla\lambda_j$$

Idea of Petrov-Galerkin Methods

- Try to obtain control on the pressure by adding
  - element-wise terms of the form $\delta_K h_K^2 \int_K \nabla p_\mathcal{T} \cdot \nabla q_\mathcal{T}$,
  - face-wise terms of the form $\delta_E h_E \int_E [p_\mathcal{T}]_E [q_\mathcal{T}]_E$.
  - The form of the scaling parameters is motivated by the Mini element and the request that element and face contributions should be of comparable size.
- The resulting problem should be coercive.
- Contrary to penalty methods, the additional terms should be consistent with the variational problem, i.e. they should vanish for the weak solution of the Stokes problem.
- Pressure-jumps are no problem.
- Test the momentum equation element-wise with $\delta_K h_K^2 \nabla q_\mathcal{T}$.

General Form of Petrov-Galerkin Methods

Find $\mathbf{u}_\mathcal{T} \in X(\mathcal{T})$, $p_\mathcal{T} \in Y(\mathcal{T})$ such that for all $\mathbf{v}_\mathcal{T} \in X(\mathcal{T})$, $q_\mathcal{T} \in Y(\mathcal{T})$

$$\int_\Omega \nabla\mathbf{u}_\mathcal{T} : \nabla\mathbf{v}_\mathcal{T} - \int_\Omega p_\mathcal{T} \operatorname{div} \mathbf{v}_\mathcal{T} = \int_\Omega \mathbf{f} \cdot \mathbf{v}_\mathcal{T}$$

$$\int_\Omega q_\mathcal{T} \operatorname{div} \mathbf{u}_\mathcal{T}$$

$$+ \sum_{K\in\mathcal{T}} \delta_K h_K^2 \int_K (-\Delta\mathbf{u}_\mathcal{T} + \nabla p_\mathcal{T}) \cdot \nabla q_\mathcal{T}$$

$$+ \sum_{E\in\mathcal{E}_\mathcal{T}} \delta_E h_E \int_E [p_\mathcal{T}]_E [q_\mathcal{T}]_E = \sum_{K\in\mathcal{T}} \delta_K h_K^2 \int_K \mathbf{f} \cdot \nabla q_\mathcal{T}$$

Choice of Stabilization Parameters

- Set

$$\delta_{\max} = \max\{\max_{K\in\mathcal{T}} \delta_K \,,\, \max_{E\in\mathcal{E}_\mathcal{T}} \delta_E\},$$

$$\delta_{\min} = \begin{cases} \min\{\min_{K\in\mathcal{T}} \delta_K \,,\, \min_{E\in\mathcal{E}_\mathcal{T}} \delta_E\} & \text{if pressures are discontinuous,} \\ \min_{K\in\mathcal{T}} \delta_K & \text{if pressures are continuous.} \end{cases}$$

- A reasonable choice of the stabilization parameters then is determined by the condition

$$\delta_{\max} \approx \delta_{\min}.$$

Choice of Spaces

- Optimal with respect to error estimates versus degrees of freedom:

$$X(\mathcal{T}) = S_0^{k,0}(\mathcal{T})^d$$
$$Y(\mathcal{T}) = \begin{cases} S^{k-1,0}(\mathcal{T}) \cap L_0^2(\Omega) & \text{continuous pressure} \\ S^{k-1,-1}(\mathcal{T}) \cap L_0^2(\Omega) & \text{discontinuous pressure} \end{cases}$$

- Equal order interpolation:

$$X(\mathcal{T}) = S_0^{k,0}(\mathcal{T})^d$$
$$Y(\mathcal{T}) = \begin{cases} S^{k,0}(\mathcal{T}) \cap L_0^2(\Omega) & \text{continuous pressure} \\ S^{k,-1}(\mathcal{T}) \cap L_0^2(\Omega) & \text{discontinuous pressure} \end{cases}$$

Mesh-Dependent Norms and (Bi-)Linear Forms

- $|||(\mathbf{u}_\mathcal{T}, p_\mathcal{T})|||_{1,\mathcal{T}} = \left\{ |\mathbf{u}_\mathcal{T}|_1^2 + \|p_\mathcal{T}\|^2 + |p_\mathcal{T}|_{1,\mathcal{T}}^2 \right\}^{\frac{1}{2}}$
- $B_\mathcal{T}((\mathbf{u}_\mathcal{T}, p_\mathcal{T}), (\mathbf{v}_\mathcal{T}, q_\mathcal{T}))$

$$= \int_\Omega \nabla \mathbf{u}_\mathcal{T} : \nabla \mathbf{v}_\mathcal{T} - \int_\Omega p_\mathcal{T} \operatorname{div} \mathbf{v}_\mathcal{T} + \int_\Omega q_\mathcal{T} \operatorname{div} \mathbf{u}_\mathcal{T}$$
$$+ \sum_{K \in \mathcal{T}} \delta_K h_K^2 \int_K (-\Delta \mathbf{u}_\mathcal{T} + \nabla p_\mathcal{T}) \cdot \nabla q_\mathcal{T}$$
$$+ \sum_{E \in \mathcal{E}} \delta_E h_E \int_E [p_\mathcal{T}]_E [q_\mathcal{T}]_E$$

- $\ell_\mathcal{T}((\mathbf{v}_\mathcal{T}, q_\mathcal{T})) = \int_\Omega \mathbf{f} \cdot \mathbf{v}_\mathcal{T} + \sum_{K \in \mathcal{T}} \delta_K h_K^2 \int_K \mathbf{f} \cdot \nabla q_\mathcal{T}$

Stability of the Petrov-Galerkin Discretization

- Assume that $\delta_{\min} > 0$ and $\delta_{\max} < \delta_0$ where δ_0 only depends on the shape parameter of $\mathcal{T}$.
- Then there is a constant $\gamma > 0$ which does not depend on $\mathcal{T}$ such that

$$\inf_{(\mathbf{u}_\mathcal{T}, p_\mathcal{T})} \sup_{(\mathbf{v}_\mathcal{T}, q_\mathcal{T})} \frac{B_\mathcal{T}((\mathbf{u}_\mathcal{T}, p_\mathcal{T}), (\mathbf{v}_\mathcal{T}, q_\mathcal{T}))}{|||(\mathbf{u}_\mathcal{T}, p_\mathcal{T})|||_{1,\mathcal{T}} |||(\mathbf{v}_\mathcal{T}, q_\mathcal{T})|||_{1,\mathcal{T}}} \geq \gamma.$$

Proof of the Stability

- Inverse estimates imply that
  $B_\mathcal{T}((\mathbf{u}_\mathcal{T}, p_\mathcal{T}), (\mathbf{u}_\mathcal{T}, p_\mathcal{T})) \geq \frac{1}{2} \delta_{\min} \left\{ |||(\mathbf{u}_\mathcal{T}, p_\mathcal{T})|||_{1,\mathcal{T}}^2 - \|p_\mathcal{T}\|^2 \right\}$
- Due to the well-posedness of the Stokes problem, there is a velocity $\mathbf{v} \in H_0^1(\mathcal{T})^d$ with $|\mathbf{v}|_1 = \|p_\mathcal{T}\|$ and $\int_\Omega p_\mathcal{T} \operatorname{div} \mathbf{v} \geq \beta \|p_\mathcal{T}\|^2$.
- The properties of $R_\mathcal{T}$ imply that

$$B_\mathcal{T}((\mathbf{u}_\mathcal{T}, p_\mathcal{T}), (R_\mathcal{T}\mathbf{v}, 0)) \geq \left(\frac{1}{4} + \beta^2\right) \|p_\mathcal{T}\|^2 - \beta^2 |||(\mathbf{u}_\mathcal{T}, p_\mathcal{T})|||_{1,\mathcal{T}}^2,$$
$$|||(R_\mathcal{T}\mathbf{v}, 0)|||_{1,\mathcal{T}} \leq c_\Omega \|p_\mathcal{T}\|.$$

- Taking the maximum of both estimates proves the stability.

Error Estimates

- There is a constant $c \approx \delta_{\max}\gamma^{-1}$ such that

$$
\begin{aligned}
&|||(\mathbf{u}-\mathbf{u}_\mathcal{T}, p-p_\mathcal{T})|||_{1,\mathcal{T}} \\
&\quad \le c \inf_{(\mathbf{v}_\mathcal{T}, q_\mathcal{T})} \Big\{ |||(\mathbf{u}-\mathbf{v}_\mathcal{T}, p-q_\mathcal{T})|||^2_{1,\mathcal{T}} \\
&\qquad + \sum_{K\in\mathcal{T}} h_K^2 \|\Delta(\mathbf{u}-\mathbf{v}_\mathcal{T})\|_K^2 \Big\}^{\frac{1}{2}}.
\end{aligned}
$$

- If $\mathbf{u} \in H^{k+1}(\Omega)^d \cap H_0^1(\Omega)^d$, $p \in H^k(\Omega) \cap L_0^2(\Omega)$, $S_0^{k,0}(\mathcal{T})^d \subset X(\mathcal{T})$ and $S^{k-1,-1}(\mathcal{T}) \cap L_0^2(\Omega) \subset Y(\mathcal{T})$ or $S^{k-1,0}(\mathcal{T}) \cap L_0^2(\Omega) \subset Y(\mathcal{T})$ then

$$
|||(\mathbf{u}-\mathbf{u}_\mathcal{T}, p-p_\mathcal{T})|||_{1,\mathcal{T}} \le c' h^k \big\{ |\mathbf{u}|_{k+1} + |p|_k \big\}.
$$

Proof of the Error Estimates

The stability and the definitions of $B_\mathcal{T}$ and $\ell_\mathcal{T}$ yield

- $$|||(\mathbf{u}-\mathbf{u}_\mathcal{T}, p-p_\mathcal{T})|||_{1,\mathcal{T}} \le |||(\mathbf{u}-\mathbf{v}_\mathcal{T}, p-q_\mathcal{T})|||_{1,\mathcal{T}} + |||(\mathbf{v}_\mathcal{T}-\mathbf{u}_\mathcal{T}, q_\mathcal{T}-p_\mathcal{T})|||_{1,\mathcal{T}}$$
- $$|||(\mathbf{v}_\mathcal{T}-\mathbf{u}_\mathcal{T}, q_\mathcal{T}-p_\mathcal{T})|||_{1,\mathcal{T}} \le \frac{1}{\gamma} \sup_{(\mathbf{w}_\mathcal{T}, r_\mathcal{T})} \frac{B_\mathcal{T}((\mathbf{v}_\mathcal{T}-\mathbf{u}_\mathcal{T}, q_\mathcal{T}-p_\mathcal{T}), (\mathbf{w}_\mathcal{T}, r_\mathcal{T}))}{|||(\mathbf{w}_\mathcal{T}, r_\mathcal{T})|||_{1,\mathcal{T}}}$$
- $$\begin{aligned} &B_\mathcal{T}((\mathbf{v}_\mathcal{T}-\mathbf{u}_\mathcal{T}, q_\mathcal{T}-p_\mathcal{T}), (\mathbf{w}_\mathcal{T}, r_\mathcal{T})) \\ &= B_\mathcal{T}((\mathbf{v}_\mathcal{T}-\mathbf{u}, q_\mathcal{T}-p), (\mathbf{w}_\mathcal{T}, r_\mathcal{T})) \\ &\le c'' \Big\{ |||(\mathbf{u}-\mathbf{v}_\mathcal{T}, p-q_\mathcal{T})|||^2_{1,\mathcal{T}} + \sum_{K\in\mathcal{T}} h_K^2 \|\Delta(\mathbf{u}-\mathbf{v}_\mathcal{T})\|_K^2 \Big\}^{\frac{1}{2}}. \end{aligned}$$

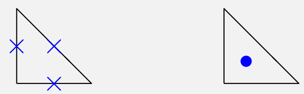
  $|||(\mathbf{w}_\mathcal{T}, r_\mathcal{T})|||_{1,\mathcal{T}}$

The Basic Idea

- We want a fully conservative discretization, i.e. the discrete solution has to satisfy $\operatorname{div} \mathbf{u}_\mathcal{T} = 0$.
- As a trade-off, we are willing to relax the conformity condition $X(\mathcal{T}) \subset H_0^1(\Omega)^d$.

The Crouzeix-Raviart Element ($d = 2$)

- $\mathcal{T}$ a triangulation
- $X(\mathcal{T}) = \{\mathbf{v}_\mathcal{T} : \mathbf{v}_\mathcal{T}|_K \in R_1(K)^2,$
  $\mathbf{v}_\mathcal{T}$ is continuous a midpoints of edges,
  $\mathbf{v}_\mathcal{T}$ vanishes at midpoints of boundary edges$\}$
- $Y(\mathcal{T}) = S^{0,-1}(\mathcal{T}) \cap L_0^2(\Omega)$
- All integrals are taken element-wise.
- Degrees of freedom:



Properties of the Crouzeix-Raviart Element

- The Crouzeix-Raviart discretization admits a unique solution $\mathbf{u}_\mathcal{T}$, $p_\mathcal{T}$.
- The discretization is fully conservative, i.e. the continuity equation $\operatorname{div} \mathbf{u}_\mathcal{T} = 0$ is satisfied element-wise.
- If Ω is convex, the following error estimates hold

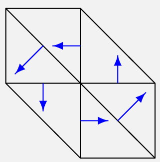
$$\Big\{ \sum_{K \in \mathcal{T}} |\mathbf{u} - \mathbf{u}_\mathcal{T}|_{1,K}^2 \Big\}^{\frac{1}{2}} + \|p - p_\mathcal{T}\| \le ch\|\mathbf{f}\|,$$
$$\|\mathbf{u} - \mathbf{u}_\mathcal{T}\| \le ch^2\|\mathbf{f}\|.$$

Drawbacks of the Crouzeix-Raviart Element

- Its accuracy deteriorates drastically in the presence of re-entrant corners.
- It has no higher order equivalent.
- It has no three-dimensional equivalent.

Construction of a Solenoidal Bases

- Denote by $\varphi_E \in S^{1,-1}(\mathcal{T})$ the function which takes the value 1 at the midpoint of E and vanishes at all other midpoints of edges.
- Set $\mathbf{w}_E = \varphi_E \mathbf{t}_E$ where $\mathbf{t}_E$ is a unit vector tangential to E.
- Set $\mathbf{w}_x = \sum_{E \in \mathcal{E}_x} \frac{1}{|E|} \varphi_E \mathbf{n}_{E,x}$.



- Then

$$V(\mathcal{T}) = \big\{ \mathbf{u}_\mathcal{T} \in X(\mathcal{T}) : \operatorname{div} \mathbf{u}_\mathcal{T} = 0 \big\}$$
$$= \operatorname{span}\big\{ \mathbf{w}_x, \mathbf{w}_E : x \in \mathcal{N}_\Omega,\, E \in \mathcal{E}_\Omega \big\}$$

Solution of the Discrete Problem

- The velocity $\mathbf{u}_\mathcal{T} \in V(\mathcal{T})$ is determined by the conditions

$$\sum_{K \in \mathcal{T}} \int_K \nabla \mathbf{u}_\mathcal{T} : \nabla \mathbf{v}_\mathcal{T} = \sum_{K \in \mathcal{T}} \int_K \mathbf{f} \cdot \mathbf{v}_\mathcal{T}$$

  for all $\mathbf{v}_\mathcal{T} \in V(\mathcal{T})$.
- The pressure $p_\mathcal{T}$ is determined by the conditions

$$\sum_{K \in \mathcal{T}} \int_K \mathbf{f} \cdot \mathbf{n}_E \varphi_E - \sum_{K \in \mathcal{T}} \int_K \nabla \mathbf{u}_\mathcal{T} : (\nabla \varphi_E \otimes \mathbf{n}_E) = -|E|[p_\mathcal{T}]_E$$

  for all $E \in \mathcal{E}_\Omega$.

Computation of the Velocity

The problem for the velocity

- is symmetric positive definite,
- corresponds to a Morley element discretization of the biharmonic equation,
- has condition number $O(h^{-4})$.

Computation of the Pressure

- Set $\mathcal{F} = \emptyset$, $\mathcal{M} = \emptyset$.
- Choose an element $K \in \mathcal{T}$ with an edge on the boundary.
  - Set $p_{\mathcal{T}} = 0$ on K.
  - Add K to $\mathcal{M}$.
- While $\mathcal{M} \neq \emptyset$ do:
  - Choose an element $K \in \mathcal{M}$.
  - For all elements K' which share an edge with K and which are not contained in $\mathcal{F}$ do:
    - On K' set $p_{\mathcal{T}}$ equal to the value of $p_{\mathcal{T}}$ on K plus the jump across the common edge.
    - If K' is not contained in $\mathcal{M}$, add it to $\mathcal{M}$.
  - Remove K from $\mathcal{M}$ and add it to $\mathcal{F}$.
- Compute the average of $p_{\mathcal{T}}$ and subtract it from $p_{\mathcal{T}}$ on every element.

The curl Operators ($d = 2$)

- $\mathbf{curl}\,\varphi = \begin{pmatrix} -\frac{\partial \varphi}{\partial x_2} \\ \frac{\partial \varphi}{\partial x_1} \end{pmatrix}$
- $\operatorname{curl} \mathbf{v} = \frac{\partial v_1}{\partial x_2} - \frac{\partial v_2}{\partial x_1}$
- $\operatorname{curl}(\mathbf{curl}\,\varphi) = -\Delta\varphi$
- $\mathbf{curl}(\operatorname{curl}\mathbf{v}) = -\Delta\mathbf{v} + \nabla(\operatorname{div}\mathbf{v})$
- $\operatorname{curl}(\nabla\varphi) = 0$
- $\operatorname{div}\mathbf{u} = 0$ if and only if there is a stream-function ψ with $\psi = 0$ on Γ and $\mathbf{u} = \mathbf{curl}\,\psi$ in Ω

Stream-Function Formulation of the Two-Dimensional Stokes Equations

Taking the curl of the momentum equation proves:

- $\mathbf{u}$ is a solution of the two-dimensional Stokes equations if and only if
- $\mathbf{u} = \mathbf{curl}\,\psi$ and ψ solves the biharmonic equation

$$\begin{aligned} \Delta^2\psi &= \operatorname{curl}\mathbf{f} && \text{in } \Omega \\ \psi &= 0 && \text{on } \Gamma \\ \frac{\partial\psi}{\partial\mathbf{n}} &= 0 && \text{on } \Gamma \end{aligned}$$

Drawbacks of the Stream-Function Formulation

- It is restricted to two dimensions.
- It gives no information on the pressure.
- A conforming discretization of the biharmonic equation requires C^1-elements.
- Low order non-conforming discretizations of the biharmonic equation are equivalent to the Crouzeix-Raviart discretization.
- Mixed formulations of the biharmonic equation using the vorticity $\omega = \operatorname{curl} \mathbf{u}$ as additional variable are at least as difficult to discretize as the original Stokes problem.

Solution of the Discrete Problems

- Motivation
- Uzawa Type Algorithms
- Multigrid Algorithms
- Subspace Decomposition Methods
- Conjugate Gradient Type Algorithms

Direct Solvers

- Typically require $O(N^{2-\frac{1}{d}})$ storage for a discrete problem with N unknowns.
- Typically require $O(N^{3-\frac{2}{d}})$ operations.
- Yield the exact solution of the discrete problem up to rounding errors.
- Yield an approximation for the differential equation with an $O(h^\alpha) = O(N^{-\frac{\alpha}{d}})$ error (typically: $\alpha \in \{1, 2\}$).

Iterative Solvers

- Typically require $O(N)$ storage.
- Typically require $O(N)$ operations per iteration.
- Their convergence rate deteriorates with an increasing condition number of the discrete problem which typically is $O(h^{-2}) = O(N^{\frac{2}{d}})$.
- In order to reduce an initial error by a factor 0.1 one typically needs the following numbers of operations:
  - $O(N^{1+\frac{2}{d}})$ with the Gauß-Seidel algorithm,
  - $O(N^{1+\frac{1}{d}})$ with the conjugate gradient (CG-) algorithm,
  - $O(N^{1+\frac{1}{2d}})$ with the CG-algorithm with Gauß-Seidel preconditioning,
  - $O(N)$ with a multigrid (MG-) algorithm.

Nested Grids

- Often one has to solve a sequence of discrete problems $L_k u_k = f_k$ corresponding to increasingly more accurate discretizations.
- Typically there is a natural interpolation operator $I_{k-1,k}$ which maps functions associated with the $(k-1)$-st discrete problem into those corresponding to the k-th discrete problem.
- Then the interpolate of any reasonable approximate solution of the $(k-1)$-st discrete problem is a good initial guess for any iterative solver applied to the k-th discrete problem.
- Often it suffices to reduce the initial error by a factor 0.1.

Nested Iteration

- Compute
$$\widetilde{u}_0 = u_0 = L_0^{-1} f_0.$$
- For $k = 1, \ldots$ compute an approximate solution $\widetilde{u}_k$ for $u_k = L_k^{-1} f_k$ by applying m_k iterations of an iterative solver for the problem
$$L_k u_k = f_k$$
with starting value $I_{k-1,k}\widetilde{u}_{k-1}$.
- m_k is implicitly determined by the stopping criterion
$$\| f_k - L_k \widetilde{u}_k \| \le \varepsilon \| f_k - L_k(I_{k-1,k}\widetilde{u}_{k-1})\|.$$

Structure of Discrete Stokes Problems

Discrete Stokes problems have the form $\begin{pmatrix} A & B \\ B^T & -\delta C \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ p \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ \delta g \end{pmatrix}$ with:

- $\delta = 0$ for mixed methods,
- $0 < \delta \approx 1$ for Petrov-Galerkin methods,
- a square, symmetric, positive definite $n_{\mathbf{u}} \times n_{\mathbf{u}}$ matrix A with condition of $O(h^{-2})$,
- a rectangular $n_{\mathbf{u}} \times n_p$ matrix B,
- a square, symmetric, positive definite $n_p \times n_p$ matrix C with condition of $O(1)$,
- a vector $\mathbf{f}$ of dimension $n_{\mathbf{u}}$ discretizing the exterior force,
- a vector g of dimension n_p which equals 0 for mixed methods.

Consequences

- The stiffness matrix $\begin{pmatrix} A & B \\ B^T & -\delta C \end{pmatrix}$ is symmetric but indefinite, i.e. it has positive and negative real eigenvalues.
- Hence, standard iterative methods such as the Gauß-Seidel and CG-algorithms fail.

The Uzawa Algorithm

0. Given: an initial guess p_0, a tolerance $\varepsilon > 0$ and a relaxation parameter $\omega > 0$.

1. Set $i = 0$.

2. Apply a few Gauß-Seidel iterations to the linear system

$$A\mathbf{u} = \mathbf{f} - Bp_i$$

and denote the result by $\mathbf{u}_{i+1}$. Compute

$$p_{i+1} = p_i + \omega\{B^T\mathbf{u}_{i+1} - \delta g - \delta C p_i\}.$$

3. If

$$\|A\mathbf{u}_{i+1} + Bp_{i+1} - \mathbf{f}\| + \|B^T\mathbf{u}_{i+1} - \delta C p_{i+1} - \delta g\| \leq \varepsilon$$

return $\mathbf{u}_{i+1}$ and p_{i+1} as approximate solution; stop. Otherwise increase i by 1 and go to step **2**.

Properties of the Uzawa Algorithm

- $\omega \in (1, 2)$, typically $\omega = 1.5$.
- Typically $\|\mathbf{v}\| = \sqrt{\frac{1}{n_\mathbf{u}}\mathbf{v}\cdot\mathbf{v}}$ and $\|q\| = \sqrt{\frac{1}{n_p} q \cdot q}$.
- The problem $A\mathbf{u} = \mathbf{f} - Bp_i$ is a discrete version of d Poisson equations for the components of the velocity field.
- The Uzawa algorithm falls into the class of pressure correction schemes.
- The convergence rate of the Uzawa algorithm is $1 - O(h^2)$.

Idea for an Improvement of the Uzawa Algorithm

- The problem $\begin{pmatrix} A & B \\ B^T & -\delta C \end{pmatrix}\begin{pmatrix} \mathbf{u} \\ p \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ \delta g \end{pmatrix}$ is equivalent to $\mathbf{u} = A^{-1}(\mathbf{f} - Bp)$ and $B^T A^{-1}(\mathbf{f} - Bp) - \delta C p = \delta g$.
- The matrix $B^T A^{-1} B + \delta C$ is symmetric, positive definite and has a condition of $O(1)$.
- Hence, a standard CG-algorithm can be applied to the pressure problem and has a uniform convergence rate independently of any mesh-size.
- The evaluation of $A^{-1}\mathbf{g}$ corresponds to the solution of d discrete Poisson equations $A\mathbf{u} = \mathbf{g}$ for the components of $\mathbf{u}$.
- The discrete Poisson problems can efficiently be solved with a MG-algorithm.

Properties of $B^T A^{-1} B$

- Identify vectors with corresponding finite element functions.
- $\mathbf{u} = A^{-1}Bp$ satisfies:
  - $\int_\Omega \nabla\mathbf{u} : \nabla\mathbf{v} = \int_\Omega p \operatorname{div}\mathbf{v}$ for all $\mathbf{v}$,
  - $|\mathbf{u}|_1 \leq \sqrt{d}\|p\|$,
  - $|\mathbf{u}|_1 \geq \beta\|p\|$ (inf-sup condition).
- $q = B^T\mathbf{u} = B^T A^{-1} B p$ satisfies:
  - $\int_\Omega qr = \int_\Omega r \operatorname{div}\mathbf{u}$ for all r,
  - $\|q\| \leq \sqrt{d}|\mathbf{u}|_1 \leq d\|p\|$,
  - $\beta^2\|p\|^2 \leq |\mathbf{u}|_1^2 = \int_\Omega p \operatorname{div}\mathbf{u} = \int_\Omega qp \leq \|q\|\|p\|$.

The Improved Uzawa Algorithm

0. Given: an initial guess p_0 and a tolerance $\varepsilon > 0$.

1. Apply a MG-algorithm with starting value zero and tolerance ε to $A\mathbf{v} = \mathbf{f} - Bp_0$ and denote the result by $\mathbf{u}_0$. Compute $r_0 = B^T\mathbf{u}_0 - \delta g - \delta C p_0$, $d_0 = r_0$, $\gamma_0 = r_0 \cdot r_0$. Set $\mathbf{u}_0 = 0$ and $i = 0$.

2. If $\gamma_i < \varepsilon^2$ compute $p = p_0 + p_i$, apply a MG-algorithm with starting value zero and tolerance ε to $A\mathbf{v} = \mathbf{f} - Bp$ and denote the result by $\mathbf{u}$, stop.

3. Apply a MG-algorithm with starting value $\mathbf{u}_i$ and tolerance ε to $A\mathbf{v} = Bd_i$ and denote the result by $\mathbf{u}_{i+1}$. Compute $s_i = B^T\mathbf{u}_{i+1} + \delta C d_i$, $\alpha_i = \frac{\gamma_i}{d_i \cdot s_i}$, $p_{i+1} = p_i + \alpha_i d_i$, $r_{i+1} = r_i - \alpha_i s_i$, $\gamma_{i+1} = r_{i+1} \cdot r_{i+1}$, $\beta_i = \frac{\gamma_{i+1}}{\gamma_i}$, $d_{i+1} = r_{i+1} + \beta_i d_i$. Increase i by 1 and go to step **2**.

Properties of the Improved Uzawa Algorithm

- It is a nested iteration with MG-iterations in the inner loops.
- Typically 2 to 4 MG-iterations suffice in the inner loops.
- It requires $O(N)$ operations per iteration.
- Its convergence rate is uniformly less than 1 for all meshes.
- It yields an approximate solution with error less than ε with $O(N \ln \varepsilon)$ operations.
- Numerical experiments yield convergence rates less than 0.5.

Convergence Rate of the Improved Uzawa Algorithm

- Denote by $M\mathbf{v}$ the result of the MG-algorithm applied to a problem with right-hand side $\mathbf{v}$.
- The improved Uzawa algorithm then corresponds to a CG-algorithm applied to the problem $B^T M(\mathbf{f} - Bp) - \delta C p = \delta g$.
- Properties of the MG-algorithm imply that
  - M is symmetric,
  - M satisfies $\|M\mathbf{v} - A^{-1}\mathbf{v}\| \le \varepsilon \|A^{-1}\mathbf{v}\|$ for all $\mathbf{v}$.
- Hence, $(1-\varepsilon)p^T B^T A^{-1} Bp \le p^T B^T M B p \le (1+\varepsilon) p^T B^T A^{-1} Bp$ for all p.
- Thus, $B^T M B$ is symmetric, positive definite and has a condition of $O(1)$ uniformly for all meshes.

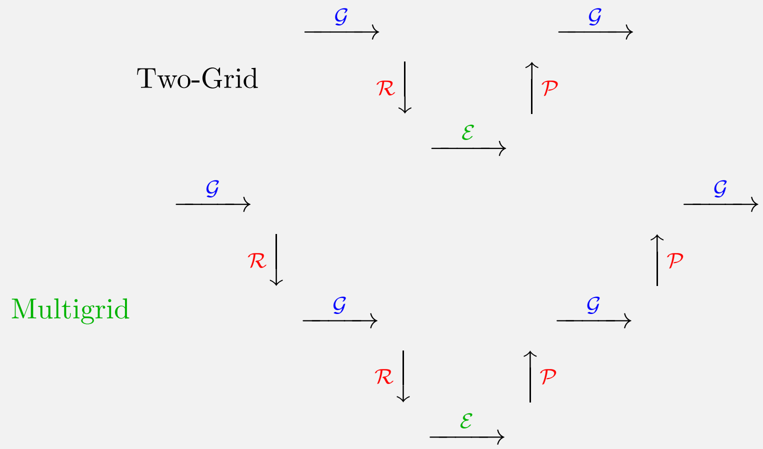
The Basic Idea

- Classical iterative methods such as the Gauß-Seidel algorithm quickly reduce highly oscillatory error components.
- Classical iterative methods such as the Gauß-Seidel algorithm are very poor in reducing slowly oscillatory error components.
- Slowly oscillating error components can well be resolved on coarser meshes with fewer unknowns.

The Basic Two-Grid Algorithm

- Perform several steps of a classical iterative method on the current grid.
- Correct the current approximation as follows:
  - Compute the current residual.
  - Restrict the residual to the next coarser grid.
  - Exactly solve the resulting problem on the coarse grid.
  - Prolongate the coarse-grid solution to the next finer grid.
- Perform several steps of a classical iterative method on the current grid.

Schematic Form



Basic Ingredients

- A sequence $\mathcal{T}_k$ of increasingly refined meshes with associated discrete problems $L_k u_k = f_k$.
- A smoothing operator M_k, which should be easy to evaluate and which at the same time should give a reasonable approximation to L_k^{-1}.
- A restriction operator $R_{k,k-1}$, which maps functions on a fine mesh $\mathcal{T}_k$ to the next coarser mesh $\mathcal{T}_{k-1}$.
- A prolongation operator $I_{k-1,k}$, which maps functions from a coarse mesh $\mathcal{T}_{k-1}$ to the next finer mesh $\mathcal{T}_k$.

The Multigrid Algorithm

0. Given: the actual level k, parameters μ, ν_1, and ν_2, the matrix L_k, the right-hand side f_k, an initial guess u_k. Sought: improved approximate solution u_k.

1. If $k = 0$ compute $u_0 = L_0^{-1} f_0$; stop.

2. (Pre-smoothing) Perform ν_1 steps of the iterative procedure $u_k \mapsto u_k + M_k(f_k - L_k u_k)$.

3. (Coarse grid correction)

- **3.1** Compute $f_{k-1} = R_{k,k-1}(f_k - L_k u_k)$ and set $u_{k-1} = 0$.
- **3.2** Perform μ iterations of the MG-algorithm with parameters $k-1$, μ, ν_1, ν_2, L_{k-1}, f_{k-1}, u_{k-1} and denote the result by u_{k-1}.
- **3.3** Update u_k by $u_k \mapsto u_k + I_{k-1,k} u_{k-1}$.

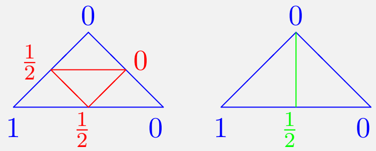
4. (Post-smoothing) Perform ν_2 steps of the iterative procedure $u_k \mapsto u_k + M_k(f_k - L_k u_k)$.

Typical Choices of Parameters

- $\mu = 1$ V-cycle or
  $\mu = 2$ W-cycle
- $\nu_1 = \nu_2 = \nu$ or
  $\nu_1 = \nu$, $\nu_2 = 0$ or
  $\nu_1 = 0$, $\nu_2 = \nu$
- $1 \leq \nu \leq 4$.

Prolongation and Restriction

- The prolongation is typically determined by the natural inclusion of the finite element spaces, i.e. a finite element function corresponding to a coarse mesh is expressed in terms of the finite element bases functions corresponding to the fine mesh.



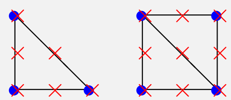
- The restriction is typically determined by inserting finite element bases functions corresponding to the coarse mesh in the variational form of the discrete problem corresponding to the fine mesh.

Smoothing (Positive Definite Problems)

- Gauß-Seidel iteration
- SSOR iteration:
  - Perform a forward Gauß-Seidel sweep with over-relaxation as pre-smoothing.
  - Perform a backward Gauß-Seidel sweep with over-relaxation as post-smoothing.
- ILU smoothing:
  - Perform an incomplete lower upper decomposition of L_k by suppressing all fill-in.
  - The result is an approximate decomposition $\mathcal{L}_k \mathcal{U}_k \approx L_k$.
  - Compute $v_k = M_k u_k$ by solving the system $\mathcal{L}_k \mathcal{U}_k v_k = u_k$.

Smoothing (Stokes Problem)

- Squared Jacobi iteration:
  - $M_k = \frac{1}{\omega^2} \begin{pmatrix} A & h_k^{-2} B \\ h_k^{-2} B^T & -h_k^{-4} \delta C \end{pmatrix}$
  - The factors h_k^{-2} and h_k^{-4} compensate the different order of differentiation for the velocity and pressure.
- Vanka smoothers:
  - Similarly to the Gauß-Seidel iteration, simultaneously adjust all degrees of freedom for the velocity and pressure corresponding to an element or to a patch of elements while fixing the remaining degrees of freedom.
  - Patches typically consist of two elements sharing a common face or the elements sharing a given vertex.



Number of Operations

- Assume that
  - one smoothing step requires $O(N_k)$ operations,
  - the prolongation requires $O(N_k)$ operations,
  - the restriction requires $O(N_k)$ operations,
  - $\mu \leq 2$,
  - $N_k > \mu N_{k-1}$,
- then one iteration of the multigrid algorithm requires $O(N_k)$ operations.

Convergence Rate (Positive Definite Problems)

- The convergence rate is uniformly less than 1 for all meshes.
- The convergence rate is bounded by $\frac{c}{c+\nu_1+\nu_2}$ with a constant which only depends on the shape parameter of the meshes.
- Numerical experiments yield convergence rates less than 0.1.

Convergence Rate (Stokes Problem)

- The convergence rate is uniformly less than 1 for all meshes.
- The convergence rate is bounded by $\frac{c}{\sqrt{\nu_1+\nu_2}}$ with a constant which only depends on the shape parameter of the meshes.
- Numerical experiments yield convergence rates less than 0.5.

Techniques for Proving the Convergence of Multigrid Algorithms

- Methods of linear algebra and discrete Fourier analysis (Hackbusch)
- Spectral decomposition and scales of discrete Sobolev spaces (Bank-DuPont and Braess-Hackbusch)
- Subspace decomposition methods (Bramble-Pasciak-Xu and Wang)

Convergence Proof à la Hackbusch

- The iteration matrix of the smoother is $N_k = I - M_k L_k$.
- The iteration matrix of the two-grid algorithm is $\widehat{S}_k = N_k^{\nu_2}(I - I_{k-1,k} L_{k-1}^{-1} R_{k,k-1}) N_k^{\nu_1}$.
- The iteration matrix of the multigrid algorithm is $S_k = \widehat{S}_k + N_k^{\nu_2} I_{k-1,k} S_{k-1}^{\mu} L_{k-1}^{-1} R_{k,k-1} N_k^{\nu_1}$.
- Prove the smoothing property $L_k N_k^{\nu} \le \eta(\nu) h_k^{-\alpha}$ with $\eta(\nu) \to 0$ for $\nu \to \infty$ and $\alpha \ge 0$.
- Prove the approximation property $|||L_k^{-1} - I_{k-1,k} L_{k-1}^{-1} R_{k,k-1}||| \le c h^{\alpha}$.
- The smoothing and approximation property imply $|||\widehat{S}_k||| \le c\eta(\nu_1 + \nu_2)$.
- If $\mu \ge 2$ a perturbation argument yields $|||S_k||| \le 2c\eta(\nu_1 + \nu_2)$.

Establishing the Smoothing and Approximation Property

- The proof of the smoothing property is usually based on a spectral decomposition of L_k and N_k.
- The proof of the approximation property is usually based on arguments used in the proof of a priori error estimates.
- The crucial point is to correctly link both techniques.

Convergence Proof à la Braess-Hackbusch

- Assume that L_k is symmetric, positive definite and set $|||v_k||| = (v_k, L_k v_k)^{\frac{1}{2}}$.
- Denote by $Q_k = I_{k-1,k} L_{k-1}^{-1} R_{k,k-1} L_k$ the Ritz projection.
- Denote by J_k the iteration matrix of the Jacobi iteration and set $|v_k| = |||J_k^{\frac{1}{2}} v_k|||$ and $\rho(v_k) = \frac{|v_k|^2}{|||v_k|||^2}$.
- Prove that
  - $|||J_k^{\nu} v_k||| \le \rho^{\nu} |||v_k|||$ with $\rho = \rho(J_k^{\nu} v_k)$,
  - $|||v_k - Q_k v_k||| \le \min\{1, c\sqrt{1 - \rho(v_k)}\} |||v_k|||$.
- Then the convergence rate δ_k of the multigrid algorithm with $\mu = 1$ and $\nu_1 = \nu_2 = \nu$ and Jacobi smoothing satisfies $\delta_k \le \max_{0 \le \rho \le 1} \Big\{ \rho^{2\nu} \big[\delta_{k-1} + (1 - \delta_{k-1}) \min\{1, c(1-\rho)\} \big] \Big\}$.
- By induction this proves $\delta_k \le \frac{c}{c + 2\nu}$.

The Setting

- V a finite dimensional Hilbert space with inner product $(\cdot, \cdot)$
- V_i, $1 \le i \le N$, subspaces of V with $\sum_i V_i = V$
  The decomposition usually neither is direct nor orthogonal.
- $Q_i : V \to V_i$ orthogonal projection w.r.t. to $(\cdot, \cdot)$
- $A : v \to V$ a symmetric, positive definite operator
- $A_i : V_i \to V_i$ the restriction of A to V_i
- $R_i : V_i \to V_i$ an easy-to-evaluate, symmetric, positive definite approximation to A_i^{-1}

The Subspace Decomposition Algorithm

- Given an initial guess $u_0 \in V$.
- For $n = 0, 1, \ldots$ and $j = 1, \ldots, N$ compute

$$u_{n+\frac{j}{N}} = u_{n+\frac{j-1}{N}} + R_j Q_j (f - A u_{n+\frac{j-1}{N}}).$$

Examples

- $V = \mathbb{R}^N$ and $V_i = \operatorname{span}\{e_i\}$ corresponds to the classical Gauß-Seidel algorithm.
- $V = S_0^{k,0}(\mathcal{T}_N)$, $V_i = S_0^{k,0}(\mathcal{T}_i)$, $R_0 = A_0^{-1}$ and $R_i = \frac{1}{\omega_i} I$ with uniformly or locally refined nested meshes $\mathcal{T}_i$ corresponds to the multigrid algorithm with Jacobi smoothing and $\nu_1 = 1$, $\nu_2 = 0$.

Convergence Rate

- Set $|||v||| = (Av, v)^{\frac{1}{2}}$.
- Set $\lambda = \min_i \lambda_{\max}(R_i A_i)$ and $\Lambda = \max_i \lambda_{\max}(R_i A_i)$.
- Assume that $\Lambda < 2$.
- Assume that there are two constants K_0 and K_1 such that
  - $\left\{\sum_{i=1}^{N} |||v_i|||^2\right\}^{\frac{1}{2}} \le K_0 |||v|||$ for all $v = \sum_i v_i$,
  - $\sum_{1 \le i,j \le N} (A v_i, w_j) \le K_1 \left\{\sum_{i=1}^{N} |||v_i|||^2\right\}^{\frac{1}{2}} \left\{\sum_{j=1}^{N} |||w_j|||^2\right\}^{\frac{1}{2}}$ for all $v_i \in V_i$, $w_j \in V_j$.
- Then the convergence rate of the subspace decomposition algorithm w.r.t. $|||\cdot|||$ is less than $\left[1 - \left(\frac{2}{\Lambda} - 1\right)\left(\frac{\lambda}{\Lambda K_0 K_1}\right)^2\right]^{\frac{1}{2}}$.

Proof of the Bound for the Convergence Rate

- Set $T_i = R_i Q_i A$, $E_0 = I$, $E_j = (I - T_j) \cdot \ldots \cdot (I - T_1)$.
- For every j this gives $-E_j + E_{j-1} = T_j E_{j-1}$ and $|||E_{j-1} v|||^2 - |||E_j v|||^2 = (A T_j E_{j-1} v, (2I - T_j) E_{j-1} v) \ge (2 - \Lambda)(A T_j E_{j-1} v, E_{j-1} v)$.
- Summation yields $|||v|||^2 - |||E_N v|||^2 \ge (2 - \Lambda) \sum_j (A T_j E_{j-1} v, E_{j-1} v)$.
- The first assumption yields for $v = \sum_i v_i$

$$|||v|||^2 = \textstyle\sum_i (A (R_i A_i)^{-1} T_i v, v) \le \lambda^{-1} K_0 |||v||| \left\{\sum_i |||T_i v|||^2\right\}^{\frac{1}{2}}.$$

- The second assumption implies $\sum_i |||T_i v|||^2 \le \Lambda^3 K_1^2 \sum_j (A T_j E_{j-1} v, E_{j-1} v)$.
- Combining all estimates yields the bound for the convergence rate.

Verification of the Assumptions

- The condition $\Lambda < 2$ can be satisfied by a suitable scaling of R_i.
- Since $V = \sum_i V_i$ the mapping $V_1 \times \ldots \times V_N \ni (v_1, \ldots, v_N) \mapsto \sum_i v_i \in V$ is surjective. The open mapping theorem therefore proves the first assumption. The crucial point is to obtain an explicit bound for K_0 which does not depend on N. This requires deep results concerning the characterization of Sobolev spaces as approximation spaces.
- Due to the Cauchy-Schwarz inequality, the second assumption is always satisfied with $K_1 \leq N$. If the subspaces satisfy a strengthened Cauchy-Schwarz inequality $(Av_i, w_j) \leq \gamma^{|i-j|} |||v_i||| |||w_j|||$ with $\gamma < 1$, the second assumption is satisfied with $K_1 \leq \frac{1}{1-\gamma}$.

CG-Type Algorithms for Non-Symmetric and Indefinite Systems of Equations

- The classical CG-algorithm breaks down for non-symmetric or indefinite systems of equations.
- A naive remedy is to apply the CG-algorithm to the system $L^T Lu = L^T f$ of the normal equations.
- This approach cannot be recommended since passing to the normal system squares the condition number.
- The following variants of the CG-algorithm are particularly adapted to non-symmetric and indefinite problems:
  - the stabilized bi-conjugate gradient algorithm (Bi-CG-stab in short),
  - the generalized minimal residual method (GMRES in short).

The Idea of the Bi-CG-stab Algorithm

- The algorithm tries to simultaneously solve the original problem $Lu = f$ and its adjoint problem $L^T v = f$.
- For both problems it performs a simultaneous three-term recursion similar to the CG-iteration.
- It incorporates particular devices to detect possible break-downs and to restart the iteration before breaking down.
- It can be combined with preconditioning. Possible methods for preconditioning are:
  - the SSOR-iteration applied to the symmetric part of L,
  - incomplete factorizations of L as used in the context of ILU-smoothing.

The Idea of the GMRES Algorithm

- It performs a three-term recursion to build increasingly larger Krylov spaces $K_n = \text{span}\{u, Lu, \ldots, L^{n-1}u\}$.
- For every Krylov space K_n it approximately solves the minimization problem $v_n = \text{argmin}_{w \in K_n} \|Lw - f\|$ using a QR-method.

A Posteriori Error Estimation and Adaptivity

- Motivation
- A Posteriori Error Estimates for the Stokes Problem
- Mesh Refinement, Coarsening and Smoothing

Drawbacks of A Priori Error Estimates

- They only yield information on the asymptotic behaviour of the error.
- They require regularity properties of the solution which often are not realistic.
- They give no information on the actual size of the error.
- They are not able to detect local singularities arising from re-entrant corners or boundary or interior layers which deteriorate the overall accuracy of the discretization.

Goal of A Posteriori Error Estimation and Adaptivity

- We want to obtain explicit information about the error of the discretization and its spatial (and temporal) distribution.
- The information should a posteriori be extracted from the computed numerical solution and the given data of the problem.
- The cost for obtaining this information should be far less than for the computation of the numerical solution.
- We want to obtain a numerical solution with a prescribed tolerance using a (nearly) minimal number of grid-points.
- To this end we need reliable upper and lower bounds for the true error in a user-specified norm.

General Adaptive Algorithm

0. Given: The data of a partial differential equation and a tolerance ε.
   Sought: A numerical solution with an error less than ε.
1. Construct an initial coarse mesh $\mathcal{T}_0$ representing sufficiently well the geometry and data of the problem; set $k = 0$.
2. Solve the discrete problem on $\mathcal{T}_k$.
3. For every element K in $\mathcal{T}_k$ compute an a posteriori error indicator.
4. If the estimated global error is less than ε then stop. Otherwise decide which elements have to be refined or coarsened and construct the next mesh $\mathcal{T}_{k+1}$. Replace k by $k + 1$ and return to step **2**.

Basic Ingredients

- An error indicator which furnishes the a posteriori error estimate.
- A refinement strategy which determines which elements have to be refined or coarsened and how this has to be done.

The Stokes Problem and its Discretization

- $\mathbf{u} \in H_0^1(\Omega)^d$, $p \in L_0^2(\Omega)$ weak solution of the Stokes problem with no-slip boundary condition:

$$\begin{aligned} -\Delta \mathbf{u} + \operatorname{grad} p = \mathbf{f} &\quad \text{in } \Omega \\ \operatorname{div} \mathbf{u} = 0 &\quad \text{in } \Omega \\ \mathbf{u} = 0 &\quad \text{on } \Gamma \end{aligned}$$

- $\mathbf{u}_\mathcal{T} \in X(\mathcal{T})$, $p_\mathcal{T} \in Y(\mathcal{T})$ solution of a mixed or Petrov-Galerkin discretization of the Stokes problem
- Assume that $S_0^{1,0}(\mathcal{T})^d \subset X(\mathcal{T})$

Residual

- Define two residuals $R_\mathrm{m} \in H^{-1}(\Omega)^d$ and $R_\mathrm{c} \in L^2(\Omega)$ associated with the momentum and continuity equation by

$$\begin{aligned} \langle R_\mathrm{m}, \mathbf{v} \rangle &= \int_\Omega \mathbf{f} \cdot \mathbf{v} - \int_\Omega \nabla \mathbf{u}_\mathcal{T} : \nabla \mathbf{v} + \int_\Omega p_\mathcal{T} \operatorname{div} \mathbf{v} \\ \langle R_\mathrm{c}, q \rangle &= \int_\Omega q \operatorname{div} \mathbf{u}_\mathcal{T} \end{aligned}$$

- Then the error $\mathbf{u} - \mathbf{u}_\mathcal{T}$, $p - p_\mathcal{T}$ solves the Stokes problem

$$\begin{aligned} \int_\Omega \nabla(\mathbf{u} - \mathbf{u}_\mathcal{T}) : \nabla \mathbf{v} - \int_\Omega (p - p_\mathcal{T}) \operatorname{div} \mathbf{v} &= \langle R_\mathrm{m}, \mathbf{v} \rangle \\ \int_\Omega q \operatorname{div}(\mathbf{u} - \mathbf{u}_\mathcal{T}) &= \langle R_\mathrm{c}, q \rangle \end{aligned}$$

Equivalence of Error and Residual

- The well-posedness of the saddle-point formulation of the Stokes problem implies

$$\begin{aligned} \frac{1}{c_*} \big\{ \|R_\mathrm{m}\|_{-1} + \|R_\mathrm{c}\| \big\} &\le |\mathbf{u} - \mathbf{u}_\mathcal{T}|_1 + \|p - p_\mathcal{T}\| \\ &\le c^* \big\{ \|R_\mathrm{m}\|_{-1} + \|R_\mathrm{c}\| \big\}. \end{aligned}$$

- c_* and c^* depend on the space dimension d.
- c^* in addition depends on the constant in the inf-sup condition for the Stokes problem.
- The above equivalence holds for every discretization be it stable or not.

Evaluation of $\|R_\mathbf{c}\|$

- The definition of R_c implies

$$\|R_\mathrm{c}\| = \|\operatorname{div} \mathbf{u}_\mathcal{T}\|.$$

- Hence, $\|R_\mathrm{c}\|$ can be evaluated easily and is a measure for the lacking incompressibility of $\mathbf{u}_\mathcal{T}$.

Evaluation of $\|R_\mathbf{m}\|_{-1}$

- The explicit evaluation of $\|R_\mathrm{m}\|_{-1}$ would require the solution of an infinite dimensional variational problem which is as expensive as the solution of the original Stokes problem.
- Hence, we must obtain estimates for $\|R_\mathrm{m}\|_{-1}$ which at the same time are as sharp as possible and easy to evaluate.
- Main tools for achieving this goal are:
  - properties of the discrete problem,
  - the Galerkin orthogonality of R_m,
  - an L^2-representation of R_m,
  - approximation properties of the quasi-interpolation operator $R_\mathcal{T}$,
  - inverse estimates for the bubble functions.

The Discrete Problem Reviewed

$\mathbf{u}_\mathcal{T}$, $p_\mathcal{T}$ satisfies for every $\mathbf{v}_\mathcal{T} \in X(\mathcal{T})$, $q_\mathcal{T} \in Y(\mathcal{T})$

$$\begin{aligned}
0 &= \ell_\mathcal{T}((\mathbf{v}_\mathcal{T}, q_\mathcal{T})) - B_\mathcal{T}((\mathbf{u}_\mathcal{T}, p_\mathcal{T}), (\mathbf{v}_\mathcal{T}, q_\mathcal{T})) \\
&= \underbrace{\int_\Omega \mathbf{f}\cdot\mathbf{v}_\mathcal{T} - \int_\Omega \nabla\mathbf{u}_\mathcal{T} : \nabla\mathbf{v}_\mathcal{T} + \int_\Omega p_\mathcal{T}\operatorname{div}\mathbf{v}_\mathcal{T}}_{=\langle R_\mathrm{m}, \mathbf{v}_\mathcal{T}\rangle} - \int_\Omega q_\mathcal{T}\operatorname{div}\mathbf{u}_\mathcal{T} \\
&\quad + \sum_{K\in\mathcal{T}} \delta_K h_K^2 \int_K \mathbf{f}\cdot\nabla q_\mathcal{T} \\
&\quad - \sum_{K\in\mathcal{T}} \delta_K h_K^2 \int_K (-\Delta\mathbf{u}_\mathcal{T} + \nabla p_\mathcal{T})\cdot\nabla q_\mathcal{T} \\
&\quad - \sum_{E\in\mathcal{E}} \delta_E h_E \int_E [p_\mathcal{T}]_E [q_\mathcal{T}]_E
\end{aligned}$$

Galerkin Orthogonality of $R_\mathbf{m}$

The form of the discrete problem and the assumption $S_0^{1,0}(\mathcal{T})^d \subset X(\mathcal{T})$ imply the Galerkin orthogonality

$$\langle R_\mathrm{m}, \mathbf{v}_\mathcal{T}\rangle = 0 \quad \forall \mathbf{v}_\mathcal{T} \in S_0^{1,0}(\mathcal{T})^d.$$

L^2-Representation of R_{m}

Integration by parts element-wise yields for every $\mathbf{v} \in H_0^1(\Omega)^d$ the L^2-representation

$$\langle R_{\mathrm{m}}, \mathbf{v}\rangle = \sum_{K\in\mathcal{T}} \int_K \big(\mathbf{f} + \Delta \mathbf{u}_{\mathcal{T}} - \nabla p_{\mathcal{T}}\big) \cdot \mathbf{v} - \sum_{E\in\mathcal{E}_\Omega} \int_E \big[\mathbf{n}_E \cdot (\nabla \mathbf{u}_{\mathcal{T}} - p_{\mathcal{T}} I)\big]_E \cdot \mathbf{v}.$$

Upper Bounds for $\|R_{\mathrm{m}}\|_{-1}$

- The Galerkin orthogonality, the L^2-representation and the approximation properties of $R_{\mathcal{T}}$ imply

$$\begin{aligned}\langle R_{\mathrm{m}}, \mathbf{v}\rangle &= \langle R_{\mathrm{m}}, \mathbf{v} - R_{\mathcal{T}}\mathbf{v}\rangle \\ &\le \sum_{K\in\mathcal{T}} c_{A1} h_K \|\mathbf{f} + \Delta\mathbf{u}_{\mathcal{T}} - \nabla p_{\mathcal{T}}\|_K |\mathbf{v}|_{1,\widetilde{\omega}_K} \\ &\quad + \sum_{E\in\mathcal{E}_\Omega} c_{A2} h_E^{\frac{1}{2}} \|[\mathbf{n}_E \cdot (\nabla\mathbf{u}_{\mathcal{T}} - p_{\mathcal{T}} I)]_E\|_E |\mathbf{v}|_{1,\widetilde{\omega}_E}\end{aligned}$$

- The Cauchy-Schwarz inequality therefore yields

$$\|R_{\mathrm{m}}\|_{-1} \le c\Big\{\sum_{K\in\mathcal{T}} h_K^2 \|\mathbf{f} + \Delta\mathbf{u}_{\mathcal{T}} - \nabla p_{\mathcal{T}}\|_K^2 + \sum_{E\in\mathcal{E}_\Omega} h_E \|[\mathbf{n}_E\cdot(\nabla\mathbf{u}_{\mathcal{T}} - p_{\mathcal{T}} I)]_E\|_E^2\Big\}^{\frac{1}{2}}.$$

Lower Bounds for $\|R_{\mathrm{m}}\|_{-1}$

- Denote be $\mathbf{f}_{\mathcal{T}}$ any piece-wise polynomial approximation of $\mathbf{f}$.
- Inserting the functions $\psi_K(\mathbf{f}_{\mathcal{T}} + \Delta\mathbf{u}_{\mathcal{T}} - \nabla p_{\mathcal{T}})$ and $\psi_E[\mathbf{n}_E \cdot (\nabla\mathbf{u}_{\mathcal{T}} - p_{\mathcal{T}} I)]_E$ in the definition of R_{m} and using the inverse estimates for the bubble functions proves

$$h_K\|\mathbf{f}_{\mathcal{T}} + \Delta\mathbf{u}_{\mathcal{T}} - \nabla p_{\mathcal{T}}\|_K \le c\big\{\|R_{\mathrm{m}}\|_{-1,K} + h_K\|\mathbf{f} - \mathbf{f}_{\mathcal{T}}\|_K\big\}$$

$$h_E^{\frac{1}{2}}\|[\mathbf{n}_E\cdot(\nabla\mathbf{u}_{\mathcal{T}} - p_{\mathcal{T}} I)]_E\|_E \le c\big\{\|R_{\mathrm{m}}\|_{-1,\omega_E} + h_E\|\mathbf{f} - \mathbf{f}_{\mathcal{T}}\|_{\omega_E}\big\}$$

Residual A Posteriori Error Estimates

- Define the residual a posteriori error indicator $\eta_{R,K}$ by

$$\eta_{R,K} = \Big\{h_K^2\|\mathbf{f}_{\mathcal{T}} + \Delta\mathbf{u}_{\mathcal{T}} - \nabla p_{\mathcal{T}}\|_K^2 + \|\mathrm{div}\,\mathbf{u}_{\mathcal{T}}\|_K^2 + \frac{1}{2}\sum_{E\in\mathcal{E}_{K,\Omega}} h_E\|[\mathbf{n}_E\cdot(\nabla\mathbf{u}_{\mathcal{T}} - p_{\mathcal{T}} I)]_E\|_E^2\Big\}^{\frac{1}{2}}.$$

- Then the error is bounded from above and from below by

$$\big\{|\mathbf{u} - \mathbf{u}_{\mathcal{T}}|_1^2 + \|p - p_{\mathcal{T}}\|^2\big\}^{\frac{1}{2}} \le c^*\Big\{\sum_{K\in\mathcal{T}} \big(\eta_{R,K}^2 + h_K^2\|\mathbf{f} - \mathbf{f}_{\mathcal{T}}\|_K^2\big)\Big\}^{\frac{1}{2}}$$

$$\eta_{R,K} \le c_*\Big\{|\mathbf{u} - \mathbf{u}_{\mathcal{T}}|_{1,\omega_K}^2 + \|p - p_{\mathcal{T}}\|_{\omega_K}^2 + h_K^2\|\mathbf{f} - \mathbf{f}_{\mathcal{T}}\|_{\omega_K}^2\Big\}^{\frac{1}{2}}$$

Discussion of the A Posteriori Error Estimates I

- The constants c^* and c_* depend on the shape parameter of $\mathcal{T}$.
- The constant c_* in addition depends on the polynomial degrees of $\mathbf{u}_\mathcal{T}$, $p_\mathcal{T}$, and $\mathbf{f}_\mathcal{T}$.

Discussion of the A Posteriori Error Estimates II

- The first term in $\eta_{R,K}$ is related to the residual of $u_\mathcal{T}$, $p_\mathcal{T}$ with respect to the strong form of the momentum equation.
- The second term in $\eta_{R,K}$ is related to the residual of $\mathbf{u}_\mathcal{T}$ with respect to the strong form of the continuity equation.
- The third term in $\eta_{R,K}$ is related to the boundary operator which canonically links the strong and weak form of the momentum equation.
- The third term in $\eta_{R,K}$ is crucial for low order discretizations.
- The different scalings of the three terms in $\eta_{R,K}$ take into account the different order of the differential operators.

Discussion of the A Posteriori Error Estimates III

- The upper bound is global.
- This is due to the fact that it is based on the norm of the inverse of the Stokes operator which is a global operator. (local force $\longrightarrow$ global flow)
- The lower bound is local.
- This is due to the fact that it is based on the norm of the Stokes operator itself which is a local operator. (local flow $\longrightarrow$ local force)

Auxiliary Discrete Stokes Problems

- With every element $K \in \mathcal{T}$ associate
  - a patch $\mathcal{T}_K \subset \mathcal{T}$ containing K,
  - finite element spaces $X(\mathcal{T}_K)$, $Y(\mathcal{T}_K)$ on $\mathcal{T}_K$.
- Find $\mathbf{u}_K \in X(\mathcal{T}_K)$, $p_K \in Y(\mathcal{T}_K)$ such that for all $\mathbf{v}_K \in X(\mathcal{T}_K)$, $q_K \in Y(\mathcal{T}_K)$

$$\begin{aligned}&\int_K \nabla \mathbf{u}_K : \nabla \mathbf{v}_K - \int_K p_K \operatorname{div} \mathbf{v}_K + \int_K q_K \operatorname{div} \mathbf{u}_K \\ &= \int_K \{\mathbf{f} + \Delta \mathbf{u}_\mathcal{T} - \nabla p_\mathcal{T}\} \cdot \mathbf{v}_K + \int_{\partial K} [\mathbf{n}_K \cdot (\nabla \mathbf{u}_\mathcal{T} - p_\mathcal{T} \underline{\mathbf{I}})]_{\partial K} \cdot \mathbf{v}_K \\ &\quad + \int_K q_K \operatorname{div} \mathbf{u}_\mathcal{T}.\end{aligned}$$

- Set $\eta_{N,K} = \left\{ |\mathbf{u}_K|^2_{1,\mathcal{T}_K} + \|p_K\|^2_{\mathcal{T}_K} \right\}^{\frac{1}{2}}$.

Choice of Patches and Spaces

- Patches typically consist of:
  - the element itself: $\mathcal{T}_K = K$,
  - all elements sharing a face with K: $\mathcal{T}_K = \omega_K$,
  - all elements sharing a vertex with K: $\mathcal{T}_K = \widetilde{\omega}_K$.
- The spaces $X(\mathcal{T}_K)$, $Y(\mathcal{T}_K)$ typically consist of finite element functions of a sufficiently high degree, e.g.

$$X(\mathcal{T}_K) = \text{span}\{\psi_{K'}\mathbf{v}\,,\, \psi_{E'}\mathbf{w} \,:\, \mathbf{v} \in R_{k_\mathcal{T}}(K')^d\,,\, \mathbf{w} \in R_{k_\mathcal{E}}(E')^d\,,\, K' \in \mathcal{T}_K, E' \in \mathcal{E}_{\mathcal{T}_K,\Omega}\},$$
$$Y(\mathcal{T}_K) = \text{span}\{\psi_{K'}q \,:\, q \in R_{k_\mathbf{u}-1}(K')\,,\, K' \in \mathcal{T}_K\}.$$

with $k_\mathcal{T} = \max\{k_\mathbf{u} + d, k_p - 1\}$, $k_\mathcal{E} = \max\{k_\mathbf{u} - 1, k_p\}$.

Comparison of the Error Indicators

- Both indicators yield global upper and local lower bounds for the error.
- Each indicator can be bounded from above and from below by the other one.
- Both indicators well predict the spatial distribution of the error.
- Both indicators are well suited for adaptive mesh refinement.
- The evaluation of the residual indicator is less expensive.
- The indicator based on the auxiliary Stokes problems more precisely predicts the size of the error.

Overview

- The mesh refinement requires two key-ingredients:
  - a marking strategy that decides which elements should be refined,
  - refinement rules which determine the actual subdivision of a single element.
- To maintain the admissibility of the partitions, i.e. to avoid hanging nodes, the refinement process proceeds in two stages:
  - Firstly refine all those elements that are marked due to a too large value of η_K (regular refinement).
  - Secondly refine additional elements in order to eliminate the hanging nodes which are eventually created during the first stage (irregular refinement).
- The mesh refinement may eventually be combined with mesh coarsening and mesh smoothing.

Maximum Strategy for Marking

0. Given: A partition $\mathcal{T}$, error estimates η_K for the elements $K \in \mathcal{T}$, and a threshold $\theta \in (0, 1)$.
Sought: A subset $\widetilde{\mathcal{T}}$ of marked elements that should be refined.

1. Compute $\eta_{\mathcal{T},\max} = \max_{K\in\mathcal{T}} \eta_K$.

2. If $\eta_K \geq \theta\eta_{\mathcal{T},\max}$ mark K by putting it into $\widetilde{\mathcal{T}}$.

Equilibration Strategy for Marking (Bulk Chasing or Dörfler Marking)

0. Given: A partition $\mathcal{T}$, error estimates η_K for the elements $K \in \mathcal{T}$, and a threshold $\theta \in (0,1)$.
Sought: A subset $\widetilde{\mathcal{T}}$ of marked elements that should be refined.

1. Compute $\Theta_{\mathcal{T}} = \sum_{K\in\mathcal{T}} \eta_K^2$. Set $\Sigma_{\mathcal{T}} = 0$ and $\widetilde{\mathcal{T}} = \emptyset$.

2. If $\Sigma_{\mathcal{T}} \geq \theta\Theta_{\mathcal{T}}$ return $\widetilde{\mathcal{T}}$; stop. Otherwise go to step **3**.

3. Compute $\widetilde{\eta}_{\mathcal{T},\max} = \max_{K\in\mathcal{T}\setminus\widetilde{\mathcal{T}}} \eta_K$.

4. For all elements $K \in \mathcal{T}\setminus\widetilde{\mathcal{T}}$ check whether $\eta_K = \widetilde{\eta}_{\mathcal{T},\max}$. If this is the case, mark K by putting it into $\widetilde{\mathcal{T}}$ and add η_K^2 to $\Sigma_{\mathcal{T}}$. Otherwise skip K. When all elements have been checked, return to step **2**.

Comparison of the Marking Strategies

- The maximum strategy is cheaper.
- At the end of the equilibration strategy the set $\widetilde{\mathcal{T}}$ satisfies

$$\sum_{K\in\widetilde{\mathcal{T}}} \eta_K^2 \geq \theta \sum_{K\in\mathcal{T}} \eta_K^2 .$$

- Convergence proofs for adaptive finite element methods are often based on this property.

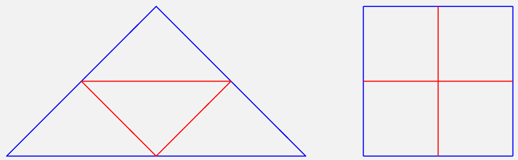
Ensuring a Sufficient Refinement

- Sometimes very few elements have an extremely large estimated error, whereas the remaining ones split into the vast majority with an extremely small estimated error and a third group of medium size consisting of elements with an estimated error of medium size.
- Then the marking strategies only refine the elements of the first group.
- This deteriorates the performance of the adaptive algorithm.
- This can be avoided by the following modification:

  Given a small percentage ε, first mark the ε% elements with largest estimated error for refinement and then apply the marking strategies to the remaining elements.

Regular Refinement

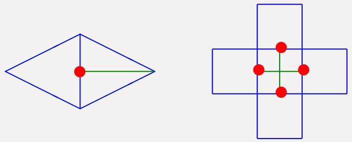
- Elements are subdivided by joining the midpoints of their edges.



- This preserves the shape parameter.

Hanging Nodes

- Hanging nodes destroy the admissibility of the partition.



- Therefore
  - either the continuity of the finite element spaces must be enforced at hanging nodes
  - or an additional irregular refinement must be performed.
- Enforcing the continuity at hanging nodes may counteract the refinement.

Computational Fluid Dynamics
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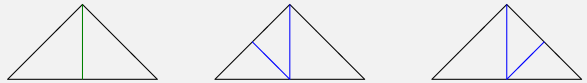
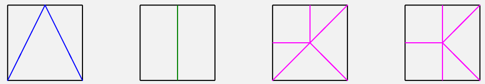
Irregular Refinement

- Triangles



- Quadrilaterals



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└Mesh Refinement, Coarsening and Smoothing
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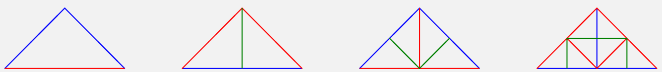
Marked Edge Bisection

- The first mesh is constructed such that the longest edge of an element is also the longest edge of its neighbour.
- The longest edges in the first mesh are marked.
- An element is refined by joining the midpoint of its marked edge with the vertex opposite to this edge (bisection).
- When bisecting the edge of an element, its two remaining edges become the marked edges of the resulting triangles.



Computational Fluid Dynamics
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Mesh Coarsening

- The coarsening of meshes is needed
  - to ensure the optimality of the adaptive process, i.e. to obtain a given accuracy with a minimal amount of unknowns,
  - to resolve moving singularities.
- The basic idea is to cluster elements with too small an error.
- This is achieved by
  - either by going back in the grid hierarchy
  - or by removing resolvable vertices.

Going Back in the Grid Hierarchy

0. Given: A hierarchy $\mathcal{T}_0, \ldots, \mathcal{T}_k$ of adaptively refined partitions, error indicators η_K for the elements K of $\mathcal{T}_k$, and parameters $1 \le m \le k$ and $n > m$.
Sought: A new partition $\mathcal{T}_{k-m+n}$.

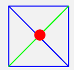
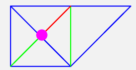
1. For every element $K \in \mathcal{T}_{k-m}$ set $\widetilde{\eta}_K = 0$.

2. For every element $K \in \mathcal{T}_k$ determine its ancestor $K' \in \mathcal{T}_{k-m}$ and add η_K^2 to $\widetilde{\eta}_{K'}^2$.

3. Successively apply the maximum or equilibration strategy n times with $\widetilde{\eta}$ as error indicator. In this process, equally distribute $\widetilde{\eta}_K$ over the descendants of K once an element K is subdivided.

Resolvable Vertices

- An element K of the current partition $\mathcal{T}$ has refinement level ℓ if it is obtained by subdividing ℓ times an element of the coarsest partition.
- Given a triangle K of the current partition $\mathcal{T}$ which is obtained by bisecting a parent triangle K', the vertex of K which is not a vertex of K' is called the refinement vertex of K.
- A vertex $z \in \mathcal{N}$ of the current partition $\mathcal{T}$ and the corresponding patch ω_z are called resolvable if
  - z is the refinement vertex of all elements contained in ω_z,
  - all elements contained in ω_z have the same refinement level.

resolvable vertex  non-resolvable vertex 

Removing Resolvable Vertices

0. Given: A partition $\mathcal{T}$, error indicators η_K for all elements K of $\mathcal{T}$, and parameters $0 < \theta_1 < \theta_2 < 1$.
Sought: Subsets $\mathcal{T}_c$ and $\mathcal{T}_r$ of elements that should be coarsened and refined, respectively.

1. Set $\mathcal{T}_c = \emptyset$, $\mathcal{T}_r = \emptyset$ and compute $\eta_{\mathcal{T},\max} = \max_{K \in \mathcal{T}} \eta_K$.

2. For all $K \in \mathcal{T}$ check whether $\eta_K \ge \theta_2 \eta_{\mathcal{T},\max}$. If this is the case, put K into $\mathcal{T}_r$.

3. For all vertices $z \in \mathcal{N}$ check whether z is resolvable. If this is the case and if $\max_{K \subset \omega_z} \eta_K \le \theta_1 \eta_{\mathcal{T},\max}$, put all elements contained in ω_z into $\mathcal{T}_c$.

Mesh Smoothing

- Improve the quality of a given partition $\mathcal{T}$ by moving its vertices while retaining the adjacency of the elements.
- The quality is measured by a a quality function q such that a larger value of q indicates a better quality.
- The quality is improved by sweeping through the vertices with a Gauß-Seidel type smoothing procedure:

For every vertex z in $\mathcal{T}$, fix the vertices of $\partial\omega_z$ and find a new vertex $\widetilde{z}$ inside ω_z such that

$$\min_{\widetilde{K} \subset \omega_{\widetilde{z}}} q(\widetilde{K}) > \min_{K \subset \omega_z} q(K).$$

Quality Functions

- Based on geometric criteria:

$$q(K) = \frac{4\sqrt{3}\mu_2(K)}{\mu_1(E_0)^2 + \mu_1(E_1)^2 + \mu_1(E_2)^2}$$

- Based on interpolation:

$$q(K) = \|\nabla(u_Q - u_L)\|_K^2$$

with linear and quadratic interpolants of u

- Based on an error indicator:

$$q(K) = \int_K \Big| \sum_{i=0}^{2} e_i \nabla \psi_{E_i} \Big|^2$$

with $e_i = h_{E_i}^{\frac{1}{2}} [\mathbf{n}_{E_i} \cdot \nabla u_{\mathcal{T}}]_{E_i}$

Stationary Incompressible Navier-Stokes Equations

- Variational Formulation
- Discretization
- Solution of the Discrete Problems
- A Posteriori Error Estimates

Strong Form

- Stationary incompressible Navier-Stokes equations in dimensionless form with no-slip boundary condition

$$\begin{aligned} -\Delta \mathbf{u} + Re(\mathbf{u}\cdot\nabla)\mathbf{u} + \operatorname{grad} p = \mathbf{f} \quad &\text{in } \Omega \\ \operatorname{div} \mathbf{u} = 0 \quad &\text{in } \Omega \\ \mathbf{u} = 0 \quad &\text{on } \Gamma \end{aligned}$$

- For the variational formulation, we want to multiply the momentum equation with a test function $\mathbf{v} \in H_0^1(\Omega)^d$ and integrate the result over Ω.
- This is possible if $\mathbf{u} \in H_0^1(\Omega)^d$ implies $(\mathbf{u}\cdot\nabla)\mathbf{u} \in H^{-1}(\Omega)^d$.

Properties of the Non-Linear Term

- Hölder's inequality yields for $\mathbf{v} \in H_0^1(\Omega)^d$, $\mathbf{u}, \mathbf{w} \in L^4(\Omega)^d$

$$\int_\Omega \big[(\mathbf{u}\cdot\nabla)\mathbf{v}\big]\cdot\mathbf{w} \le \|\mathbf{u}\|_{L^4(\Omega)} |\mathbf{v}|_1 \|\mathbf{w}\|_{L^4(\Omega)}.$$

- Since $H_0^1(\Omega)$ is continuously embedded in $L^4(\Omega)$ for $d \le 4$ (compactly for $d \le 3$), this proves that $\big[H_0^1(\Omega)^d\big]^3 \ni (\mathbf{u},\mathbf{v},\mathbf{w}) \mapsto \int_\Omega \big[(\mathbf{u}\cdot\nabla)\mathbf{v}\big]\cdot\mathbf{w}$ is a continuous trilinear form.
- Integration by parts yields for all $\mathbf{u},\mathbf{v},\mathbf{w} \in H_0^1(\Omega)^d$ with $\operatorname{div}\mathbf{u} = 0$

$$\int_\Omega \big[(\mathbf{u}\cdot\nabla)\mathbf{v}\big]\cdot\mathbf{w} = -\int_\Omega \big[(\mathbf{u}\cdot\nabla)\mathbf{w}\big]\cdot\mathbf{v}, \quad \int_\Omega \big[(\mathbf{u}\cdot\nabla)\mathbf{v}\big]\cdot\mathbf{v} = 0.$$

Variational Form

- Find $\mathbf{u} \in H_0^1(\Omega)^d$, $p \in L_0^2(\Omega)$ such that for all $\mathbf{v} \in H_0^1(\Omega)^d$, $q \in L_0^2(\Omega)$

$$\int_\Omega \nabla \mathbf{u} : \nabla \mathbf{v} - \int_\Omega p \operatorname{div} \mathbf{v} + \int_\Omega Re[(\mathbf{u} \cdot \nabla)\mathbf{u}] \cdot \mathbf{v} = \int_\Omega \mathbf{f} \cdot \mathbf{v}$$
$$\int_\Omega q \operatorname{div} \mathbf{u} = 0$$

- Equivalent form:
  Find $\mathbf{u} \in V$ such that for all $\mathbf{v} \in V$

$$\int_\Omega \nabla \mathbf{u} : \nabla \mathbf{v} + \int_\Omega Re[(\mathbf{u} \cdot \nabla)\mathbf{u}] \cdot \mathbf{v} = \int_\Omega \mathbf{f} \cdot \mathbf{v}$$

Fixed-Point Formulation

- Denote by $T : H^{-1}(\Omega)^d \to V$ the Stokes operator which associates with $\mathbf{g} \in H^{-1}(\Omega)^d$ the weak solution $T\mathbf{g} = \mathbf{v}$ of the Stokes problem with right-hand side $\mathbf{g}$, i.e.

$$\int_\Omega \nabla \mathbf{v} : \nabla \mathbf{w} = \int_\Omega \mathbf{g} \cdot \mathbf{w} \quad \forall \mathbf{w} \in V.$$

- Then the variational formulation of the Navier-Stokes equations is equivalent to

$$\mathbf{u} = T\big(\mathbf{f} - Re(\mathbf{u} \cdot \nabla)\mathbf{u}\big).$$

Properties of the Variational Problem

- Every solution satisfies the a priori bound $|\mathbf{u}|_1 \le c_F \operatorname{diam}(\Omega)\|\mathbf{f}\|$, where c_F is the constant in the Friedrichs inequality.
- If $Re\, 2^{\frac{d-1}{2}} \left[c_F \operatorname{diam}(\Omega)\right]^{3-\frac{d}{2}} \|\mathbf{f}\| < 1$, there is at most one solution.
- For every Reynolds' number Re there exists at least one solution.
- Every solution has the same regularity as the solution of the corresponding Stokes problem.
- Every solution belongs to a differentiable branch $Re \mapsto \mathbf{u}_{Re}$ of solutions which has at most a countable number of turning or bifurcation points.

A Priori Bound

Every solution $\mathbf{u}$ of the variational problem satisfies

$$\begin{aligned}
|\mathbf{u}|_1^2 &= \int_\Omega \nabla \mathbf{u} : \nabla \mathbf{u} \\
&= \int_\Omega \nabla \mathbf{u} : \nabla \mathbf{u} + Re \int_\Omega [(\mathbf{u} \cdot \nabla)\mathbf{u}] \cdot \mathbf{u} \\
&= \int_\Omega \mathbf{f} \cdot \mathbf{u} \\
&\le \|\mathbf{f}\|\|\mathbf{u}\| \\
&\le \|\mathbf{f}\| c_F \operatorname{diam}(\Omega) |\mathbf{u}|_1.
\end{aligned}$$

Uniqueness

- The difference $\mathbf{v} = \mathbf{u}_1 - \mathbf{u}_2$ of any two solutions satisfies

$$\begin{aligned}
|\mathbf{v}|_1^2 &= -Re \int_\Omega \big[(\mathbf{u}_1 \cdot \nabla)\mathbf{u}_1\big] \cdot \mathbf{v} + Re \int_\Omega \big[(\mathbf{u}_2 \cdot \nabla)\mathbf{u}_2\big] \cdot \mathbf{v} \\
&= -Re \int_\Omega \big[(\mathbf{u}_1 \cdot \nabla)\mathbf{v}\big] \cdot \mathbf{v} - Re \int_\Omega \big[(\mathbf{v} \cdot \nabla)\mathbf{u}_2\big] \cdot \mathbf{v} \\
&= -Re \int_\Omega \big[(\mathbf{v} \cdot \nabla)\mathbf{u}_2\big] \cdot \mathbf{v} \\
&\leq Re \|\mathbf{v}\|_{L^4(\Omega)}^2 |\mathbf{u}_2|_1.
\end{aligned}$$

- Combined with the a priori bound, the estimate $\|\mathbf{v}\|_{L^4(\Omega)} \leq 2^{\frac{d-1}{4}} \|\mathbf{v}\|^{1-\frac{d}{4}} |\mathbf{v}|_1^{\frac{d}{4}}$ and the Friedrichs inequality this proves the uniqueness.

Existence

- V is separable, i.e. there is as sequence of nested finite dimensional subspaces V_m such that $\bigcup_m V_m$ is dense in V.
- Denote by T_m the Stokes operator corresponding to V_m.
- The properties of the non-linear term and Schauder's fixed-point theorem imply that for every m there is a $\mathbf{u}_m \in V_m$ with $\mathbf{u}_m = T_m\big(\mathbf{f} - Re(\mathbf{u}_m \cdot \nabla)\mathbf{u}_m\big)$.
- The arguments used to prove the a priori bound imply that the sequence $\mathbf{u}_m$ is bounded in H^1.
- The compact embedding of H^1 in L^4 and the properties of the non-linear term imply that the $\mathbf{u}_m$ converge weakly to an $\mathbf{u} \in H_0^1(\Omega)^d$ which solves the variational problem with V replaced by anyone of the V_m.
- The density of $\bigcup_m V_m$ in V proves that $\mathbf{u}$ solves the variational problem.

Regularity

The regularity is proved by a bootstrap-argument using the fixed-point equation $\mathbf{u} = T\big(\mathbf{f} - Re(\mathbf{u} \cdot \nabla)\mathbf{u}\big)$ and the regularity of the Stokes problem:

$$\begin{aligned}
\mathbf{u} \in H^1 &\Longrightarrow \mathbf{u} \in L^6 && \Longrightarrow (\mathbf{u} \cdot \nabla)\mathbf{u} \in L^{\frac{5}{3}} \\
&\Longrightarrow (\mathbf{u} \cdot \nabla)\mathbf{u} \in H^{-\frac{3}{10}} && \Longrightarrow \mathbf{u} \in H^{1+\frac{7}{10}} \\
&\Longrightarrow \mathbf{u} \in L^\infty && \Longrightarrow (\mathbf{u} \cdot \nabla)\mathbf{u} \in L^2 \\
&\Longrightarrow \mathbf{u} \in H^2
\end{aligned}$$

Branches of Solutions

- The fixed-point formulation implies that there are no isolated solutions and that every solution depends in a differentiable way on Re.
- Differentiation of the fixed-point equation yields that $\mathbf{v}$, the derivative w.r.t. Re of any solution $\mathbf{u}$, satisfies

$$\mathbf{v} = -Re\, T\big((\mathbf{u} \cdot \nabla)\mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{u}\big) - T\big((\mathbf{u} \cdot \nabla)\mathbf{u}\big).$$

- The compact embedding of H^1 in L^4 implies that the operator $\mathbf{w} \mapsto T\big((\mathbf{u} \cdot \nabla)\mathbf{w} + (\mathbf{w} \cdot \nabla)\mathbf{u}\big)$ is compact.
- The statement concerning limit and bifurcation points therefore follows from properties of the spectra of compact operators, the Fredholm alternative and the a priori bound.

Basic Idea

- Replace $H_0^1(\Omega)^d$, $L_0^2(\Omega)$ by a pair $X(\mathcal{T})$, $Y(\mathcal{T})$ of finite element spaces which is uniformly stable for the Stokes problem.
- Denote by $V(\mathcal{T})$ the corresponding approximation of V.
- Since $V(\mathcal{T}) \not\subset V$ the anti-symmetry of the non-linear term is lost.
- To recover the anti-symmetry replace the non-linear term by

$$\widetilde{N}(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \frac{1}{2}\int_\Omega \left[(\mathbf{u}\cdot\nabla)\mathbf{v}\right]\cdot\mathbf{w} - \frac{1}{2}\int_\Omega \left[(\mathbf{u}\cdot\nabla)\mathbf{w}\right]\cdot\mathbf{v}.$$

Discrete Problem

- Find $\mathbf{u}_\mathcal{T} \in X(\mathcal{T})$, $p_\mathcal{T} \in Y(\mathcal{T})$ such that for all $\mathbf{v}_\mathcal{T} \in X(\mathcal{T})$, $q_\mathcal{T} \in Y(\mathcal{T})$

$$\int_\Omega \nabla\mathbf{u}_\mathcal{T} : \nabla\mathbf{v}_\mathcal{T} - \int_\Omega p_\mathcal{T} \operatorname{div}\mathbf{v}_\mathcal{T} + Re\,\widetilde{N}(\mathbf{u}_\mathcal{T}, \mathbf{u}_\mathcal{T}, \mathbf{v}_\mathcal{T}) = \int_\Omega \mathbf{f}\cdot\mathbf{v}_\mathcal{T}$$

$$\int_\Omega q_\mathcal{T} \operatorname{div}\mathbf{u}_\mathcal{T} = 0$$

- Equivalent form:
  Find $\mathbf{u}_\mathcal{T} \in V(\mathcal{T})$ such that for all $\mathbf{v}_\mathcal{T} \in V(\mathcal{T})$

$$\int_\Omega \nabla\mathbf{u}_\mathcal{T} : \nabla\mathbf{v}_\mathcal{T} + Re\,\widetilde{N}(\mathbf{u}_\mathcal{T}, \mathbf{u}_\mathcal{T}, \mathbf{v}_\mathcal{T}) = \int_\Omega \mathbf{f}\cdot\mathbf{v}_\mathcal{T}$$

Fixed-Point Formulation of the Discrete Problem

- Denote by $T_\mathcal{T} : H^{-1}(\Omega)^d \to V(\mathcal{T})$ the discrete Stokes operator which associates with $\mathbf{g} \in H^{-1}(\Omega)^d$ the weak solution $T_\mathcal{T}\mathbf{g} = \mathbf{v}_\mathcal{T}$ of the Stokes problem with right-hand side $\mathbf{g}$, i.e.

$$\int_\Omega \nabla\mathbf{v}_\mathcal{T} : \nabla\mathbf{w}_\mathcal{T} = \int_\Omega \mathbf{g}\cdot\mathbf{w}_\mathcal{T} \quad \forall \mathbf{w}_\mathcal{T} \in V(\mathcal{T}).$$

- Then the discrete problem is equivalent to

$$\mathbf{u}_\mathcal{T} = T_\mathcal{T}\big(\mathbf{f} - Re\,\widetilde{N}(\mathbf{u}_\mathcal{T}, \mathbf{u}_\mathcal{T}, \cdot)\big).$$

Properties of the Discrete Problem

- Every solution satisfies the a priori bound $|\mathbf{u}_\mathcal{T}|_1 \le c_F \operatorname{diam}(\Omega)\|\mathbf{f}\|$.
- If $Re\,2^{\frac{d-1}{2}}\left[c_F \operatorname{diam}(\Omega)\right]^{3-\frac{d}{2}}\|\mathbf{f}\| < 1$, there is at most one solution.
- For every Reynolds' number Re there exists at least one solution.
- Every solution belongs to a differentiable branch $Re \mapsto \mathbf{u}_{\mathcal{T},Re}$ of solutions which has at most a finite number of turning or bifurcation points.

Error Estimates

- Assume that:
  - $\Lambda \subset (0,\infty)$ is a compact, non empty interval.
  - $\Lambda \ni Re \mapsto \mathbf{u}_{Re}$ is a regular branch of solutions of the Navier-Stokes equations.
  - $\mathbf{u}_{Re} \in H^{k+1}(\Omega)^d$, $p_{Re} \in H^k(\Omega)$ for all $Re \in \Lambda$ with $k \ge 1$.
  - $S_0^{k,0}(\mathcal{T})^d \subset X(\mathcal{T})$ and $S^{k-1,-1}(\mathcal{T}) \cap L_0^2(\Omega) \subset Y(\mathcal{T})$ or $S^{\max\{k-1,1\},0}(\mathcal{T}) \cap L_0^2(\Omega) \subset Y(\mathcal{T})$.
- Then there is a maximal mesh-size $h_0 = h_0(\Lambda, \mathbf{f}, \mathbf{u}_{Re}) > 0$ such that for every partition $\mathcal{T}$ with $h_\mathcal{T} \le h_0$ the discrete problem has a solution $\mathbf{u}_{\mathcal{T},Re} \in X(\mathcal{T})$, $p_{\mathcal{T},Re} \in Y(\mathcal{T})$ with

$$|\mathbf{u}_{Re} - \mathbf{u}_{\mathcal{T},Re}|_1 + \|p_{Re} - p_{\mathcal{T},Re}\| \le c h_\mathcal{T}^k \sup_{Re\in\Lambda} |\mathbf{u}_{Re}|_{k+1}^2.$$

- $\|\mathbf{u}_{Re} - \mathbf{u}_{\mathcal{T},Re}\| \le c h_\mathcal{T} |\mathbf{u}_{Re} - \mathbf{u}_{\mathcal{T},Re}|_1$ if Ω is convex.

Proof of the Error Estimates. The Basic Steps

- The continuous and discrete problem can be written as $F(Re, \mathbf{u}_{Re}) = 0$ and $F_\mathcal{T}(Re, \mathbf{u}_{\mathcal{T},Re}) = 0$.
- $F_\mathcal{T}$ evaluated at the H^1-projection of $\mathbf{u}_{Re}$ is small.
- The derivative of $F_\mathcal{T}$ evaluated at the H^1-projection of $\mathbf{u}_{Re}$ is close to the derivative of F evaluated at $\mathbf{u}_{Re}$.
- The derivative of $F_\mathcal{T}$ evaluated at the H^1-projection of $\mathbf{u}_{Re}$ is invertible.
- The derivative of $F_\mathcal{T}$ is Lipschitz-continuous.
- The discrete problem has a solution in a neighbourhood of the H^1-projection of $\mathbf{u}_{Re}$.
- Compare the obtained solution with the solution of the Stokes problem with right-hand side $\mathbf{f} - Re\,(\mathbf{u}_{Re} \cdot \nabla)\mathbf{u}_{Re}$.

Proof of the Error Estimates. 1^{st} Step

Auxiliary quantities

- Define $G, \widetilde{G} \in C^\infty(H_0^1(\Omega)^d, H^{-1}(\Omega)^d)$ by

$$\langle G(\mathbf{v}), \mathbf{w}\rangle = \int_\Omega \big[(\mathbf{v}\cdot\nabla)\mathbf{v}\big]\cdot\mathbf{w}, \quad \langle \widetilde{G}(\mathbf{v}), \mathbf{w}\rangle = \widetilde{N}(\mathbf{v},\mathbf{v},\mathbf{w}).$$

- Define $F, F_\mathcal{T} \in C^\infty((0,\infty)\times H_0^1(\Omega)^d, H_0^1(\Omega)^d)$ by

$$F(Re,\mathbf{v}) = \mathbf{v} + Re\,TG(\mathbf{v}) - T\mathbf{f},$$
$$F_\mathcal{T}(Re,\mathbf{v}) = \mathbf{v} + Re\,T_\mathcal{T}\widetilde{G}(\mathbf{v}) - T_\mathcal{T}\mathbf{f}.$$

- Note that $F(Re, \mathbf{u}_{Re}) = 0$ and $F_\mathcal{T}(Re, \mathbf{u}_{\mathcal{T},Re}) = 0$.
- Denote by $\widehat{\mathbf{u}}_{\mathcal{T},Re}$ the H^1-projection of $\mathbf{u}_{Re}$ on $V(\mathcal{T})$ and set

$$\varepsilon_\mathcal{T}(Re) = |F_\mathcal{T}(Re, \widehat{\mathbf{u}}_{\mathcal{T},Re})|_1.$$

Proof of the Error Estimates. 2^{nd} Step

$$\varepsilon_\mathcal{T}(Re) \le c h_\mathcal{T}^k \sup_{Re\in\Lambda} |\mathbf{u}_{Re}|_{k+1}^2$$

- $\varepsilon_\mathcal{T}(Re) = |F_\mathcal{T}(Re, \widehat{\mathbf{u}}_{\mathcal{T},Re}) - F(Re, \mathbf{u}_{Re})|_1$
  $\le |\mathbf{u}_{Re} - \widehat{\mathbf{u}}_{\mathcal{T},Re}|_1 + Re\,|T_\mathcal{T}(\widetilde{G}(\widehat{\mathbf{u}}_{\mathcal{T},Re}) - \widetilde{G}(\mathbf{u}_{Re}))|_1$
  $+ Re\,|(T_\mathcal{T} - T)G(\mathbf{u}_{Re})|_1 + |(T - T_\mathcal{T})\mathbf{f}|_1$
- $|\mathbf{u}_{Re} - \widehat{\mathbf{u}}_{\mathcal{T},Re}|_1 \le c h_\mathcal{T}^k |\mathbf{u}_{Re}|_{k+1}$
- $|T_\mathcal{T}(\widetilde{G}(\widehat{\mathbf{u}}_{\mathcal{T},Re}) - \widetilde{G}(\mathbf{u}_{Re}))|_1 \le \|\widetilde{G}(\widehat{\mathbf{u}}_{\mathcal{T},Re}) - \widetilde{G}(\mathbf{u}_{Re})\|_{-1}$
  $\le c|\mathbf{u}_{Re}|_1 |\mathbf{u}_{Re} - \widehat{\mathbf{u}}_{\mathcal{T},Re}|_1$
- $|(T_\mathcal{T} - T)G(\mathbf{u}_{Re})|_1 \le c h_\mathcal{T}^k \|G(\mathbf{u}_{Re})\|_{k-1} \le c h_\mathcal{T}^k |\mathbf{u}_{Re}|_{k+1}^2$
- $|(T - T_\mathcal{T})\mathbf{f}|_1 \le c h_\mathcal{T}^k \|\mathbf{f}\|_{k-1} \le c h_\mathcal{T} |\mathbf{u}_{Re}|_{k+1}$

Proof of the Error Estimates. 3rd Step

$|||D_{\mathbf{v}}F_{\mathcal{T}}(Re,\widehat{\mathbf{u}}_{\mathcal{T},Re}) - D_{\mathbf{v}}F(Re,\mathbf{u}_{Re})||| \le ch|\mathbf{u}_{Re}|_2$

- $D_{\mathbf{v}}F_{\mathcal{T}}(Re,\widehat{\mathbf{u}}_{\mathcal{T},Re})\mathbf{w} - D_{\mathbf{v}}F(Re,\mathbf{u}_{Re})\mathbf{w}$
  $= Re\, T_{\mathcal{T}}(D\widetilde{G}(\widehat{\mathbf{u}}_{\mathcal{T},Re})\mathbf{w} - D\widetilde{G}(\mathbf{u}_{Re})\mathbf{w})$
  $+ Re\,(T_{\mathcal{T}} - T)D\widetilde{G}(\mathbf{u}_{Re})\mathbf{w}$
- $|T_{\mathcal{T}}(D\widetilde{G}(\widehat{\mathbf{u}}_{\mathcal{T},Re})\mathbf{w} - D\widetilde{G}(\mathbf{u}_{Re})\mathbf{w})|_1$
  $\le \|D\widetilde{G}(\widehat{\mathbf{u}}_{\mathcal{T},Re})\mathbf{w} - D\widetilde{G}(\mathbf{u}_{Re})\mathbf{w}\|_{-1}$
  $\le ch_{\mathcal{T}}|\mathbf{u}_{Re}|_2|\mathbf{w}|_1$
- $|(T_{\mathcal{T}} - T)D\widetilde{G}(\mathbf{u}_{Re})\mathbf{w}|_1 \le ch_{\mathcal{T}}\|D\widetilde{G}(\mathbf{u}_{Re})\mathbf{w}\|$
  $\le ch_{\mathcal{T}}|\mathbf{u}_{Re}|_2|\mathbf{w}|_1$

Proof of the Error Estimates. 4th Step

$|||D_{\mathbf{v}}F_{\mathcal{T}}(Re,\widehat{\mathbf{u}}_{\mathcal{T},Re})^{-1}||| \le 2|||D_{\mathbf{v}}F(Re,\mathbf{u}_{Re})^{-1}|||$ for sufficiently small $h_{\mathcal{T}}$

- $|||A||| \le \frac{1}{2}$ implies that $(I - A)$ is invertible and satisfies $|||(I - A)^{-1}||| \le 2$
- $D_{\mathbf{v}}F_{\mathcal{T}}(Re,\widehat{\mathbf{u}}_{\mathcal{T},Re})$
  $= D_{\mathbf{v}}F(Re,\mathbf{u}_{Re}) - [D_{\mathbf{v}}F(Re,\mathbf{u}_{Re}) - D_{\mathbf{v}}F_{\mathcal{T}}(Re,\widehat{\mathbf{u}}_{\mathcal{T},Re})]$
  $= D_{\mathbf{v}}F(Re,\mathbf{u}_{Re})[I -$
  $D_{\mathbf{v}}F(Re,\mathbf{u}_{Re})^{-1}(D_{\mathbf{v}}F(Re,\mathbf{u}_{Re}) - D_{\mathbf{v}}F_{\mathcal{T}}(Re,\widehat{\mathbf{u}}_{\mathcal{T},Re})]$

Proof of the Error Estimates. 5th Step

$|||D_{\mathbf{v}}F_{\mathcal{T}}(Re,\mathbf{v}_1) - D_{\mathbf{v}}F_{\mathcal{T}}(Re,\mathbf{v}_2)||| \le cRe\,|\mathbf{v}_1 - \mathbf{v}_2|_1$

- $D_{\mathbf{v}}F_{\mathcal{T}}(Re,\mathbf{v}_1)\mathbf{w} - D_{\mathbf{v}}F_{\mathcal{T}}(Re,\mathbf{v}_2)\mathbf{w}$
  $= Re\, T_{\mathcal{T}}(D\widetilde{G}(\mathbf{v}_1)\mathbf{w} - D\widetilde{G}(\mathbf{v}_2)\mathbf{w})$
- $|T_{\mathcal{T}}(D\widetilde{G}(\mathbf{v}_1)\mathbf{w} - D\widetilde{G}(\mathbf{v}_2)\mathbf{w})|_1$
  $\le \|D\widetilde{G}(\mathbf{v}_1)\mathbf{w} - D\widetilde{G}(\mathbf{v}_2)\mathbf{w}\|_{-1}$
  $\le c|\mathbf{v}_1 - \mathbf{v}_2|_1|\mathbf{w}|_1$

Proof of the Error Estimates. 6th Step

The discrete problem has a solution $\mathbf{u}_{\mathcal{T},Re}$ with $|\widehat{\mathbf{u}}_{\mathcal{T},Re} - \mathbf{u}_{\mathcal{T},Re}|_1 \le 4|||D_{\mathbf{v}}F(Re,\mathbf{u}_{Re})^{-1}|||\varepsilon_{\mathcal{T}}(Re)$

- Thanks to the fourth step we can define a mapping Φ by

$$\Phi(\mathbf{v}) = \mathbf{v} - D_{\mathbf{v}}F_{\mathcal{T}}(Re,\widehat{\mathbf{u}}_{\mathcal{T},Re})^{-1}F_{\mathcal{T}}(Re,\mathbf{v}).$$

- Then $\mathbf{u}_{\mathcal{T},Re}$ is a solution of the discrete problem if and only if it is a fixed-point of Φ.
- The fourth and fifth step imply that Φ is a contraction on the ball B in H^1 with centre $\widehat{\mathbf{u}}_{\mathcal{T},Re}$ and radius $4|||D_{\mathbf{v}}F(Re,\mathbf{u}_{Re})^{-1}|||\varepsilon_{\mathcal{T}}(Re)$.

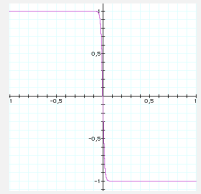
Proof of the Error Estimates. 7^{th} Step

Proof of the error estimate

- The sixth step implies that
  $|\mathbf{u}_{Re} - \mathbf{u}_{\mathcal{T},Re}|_1 \le \big(1 + 4 |||D_{\mathbf{v}} F(Re, \mathbf{u}_{Re})^{-1}|||\big) \varepsilon_{\mathcal{T}}(Re)$.
- Set $\widetilde{\mathbf{u}}_{\mathcal{T},Re} = T_{\mathcal{T}}(\mathbf{f} - \widetilde{G}(\mathbf{u}_{Re}))$ and denote by $\widetilde{p}_{\mathcal{T},Re}$ the corresponding pressure.
- The stability of the discretization implies
  $$|\mathbf{u}_{\mathcal{T},Re} - \widetilde{\mathbf{u}}_{\mathcal{T},Re}|_1 + \|p_{\mathcal{T},Re} - \widetilde{p}_{\mathcal{T},Re}\|$$
  $$\le cRe \, \|\widetilde{G}(\mathbf{u}_{Re}) - \widetilde{G}(\mathbf{u}_{\mathcal{T},Re})\|_{-1}$$
  $$\le cRe \, \big(|\mathbf{u}_{Re}|_1 + |\mathbf{u}_{\mathcal{T},Re}|_1\big) |\mathbf{u}_{Re} - \mathbf{u}_{\mathcal{T},Re}|_1.$$
- The error estimates for the Stokes problem with right-hand side $\mathbf{f} - Re\,(\mathbf{u}_{Re} \cdot \nabla)\mathbf{u}_{Re}$ yield
  $|\widetilde{\mathbf{u}}_{\mathcal{T},Re} - \mathbf{u}_{Re}|_1 + \|\widetilde{p}_{\mathcal{T},Re} - p_{Re}\| \le ch_{\mathcal{T}}^k |\mathbf{u}_{Re}|_{k+1}$.

A Warning Example

- Consider the two-point boundary value problem $-u'' + Re\, uu' = 0$ in $(-1, 1)$ with boundary conditions $u(-1) = 1$, $u(1) = -1$.
- The solution is $u(x) = -\frac{\tanh(\alpha_{Re} x)}{\tanh(\alpha_{Re})}$ where the parameter α_{Re} is determined by the relation $2\alpha_{Re} \tanh(\alpha_{Re}) = Re$.
- The solution exhibits a strong interior layer at $x = 0$.
- Explicitly solving the difference equations shows that:
  - central differences are unstable,
  - one-sided differences with a constant orientation on the whole interval are unstable,
  - one-sided differences with their orientation depending on the sign of u are stable.

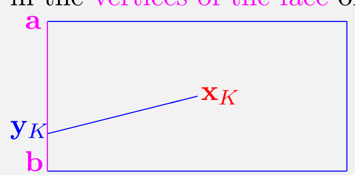


Conclusion

- We must stabilize the convective derivative.
- This can be achieved by
  - upwind schemes or
  - adding an artificial consistent viscosity in the direction of the streamlines (streamline diffusion method or SDFEM in short).

An Upwind Scheme

- Approximate the integral involving the convective derivative by a one-point quadrature rule
  $$\int_\Omega [(\mathbf{u}_{\mathcal{T}} \cdot \nabla)\mathbf{u}_{\mathcal{T}}] \cdot \mathbf{v}_{\mathcal{T}} \approx \sum_{K \in \mathcal{T}} |K| [(\mathbf{u}_{\mathcal{T}}(\mathbf{x}_K) \cdot \nabla)\mathbf{u}_{\mathcal{T}}(\mathbf{x}_K)] \cdot \mathbf{v}_{\mathcal{T}}(\mathbf{x}_K).$$
- Replace the convective derivative by an up-wind difference
  $$(\mathbf{u}_{\mathcal{T}}(\mathbf{x}_K) \cdot \nabla)\mathbf{u}_{\mathcal{T}}(\mathbf{x}_K) \approx \frac{|\mathbf{u}_{\mathcal{T}}(\mathbf{x}_K)|}{|\mathbf{x}_K - \mathbf{y}_K|}(\mathbf{u}_{\mathcal{T}}(\mathbf{x}_K) - \mathbf{u}_{\mathcal{T}}(\mathbf{y}_K)).$$
- Replace $\mathbf{u}_{\mathcal{T}}(\mathbf{y}_K)$ by $I_{\mathcal{T}}\mathbf{u}_{\mathcal{T}}(\mathbf{y}_K)$, the linear interpolate of $\mathbf{u}_{\mathcal{T}}$ in the vertices of the face of K which contains $\mathbf{y}_K$.



Drawbacks of the Upwind Scheme

- It does not fit well into the framework of variational methods.
- The discrete problem is no longer differentiable.

The Streamline Diffusion Method

Find $\mathbf{u}_\mathcal{T} \in X(\mathcal{T})$ and $p_\mathcal{T} \in Y(\mathcal{T})$ such that for all $\mathbf{v}_\mathcal{T}$, $q_\mathcal{T}$

$$\int_\Omega \nabla \mathbf{u}_\mathcal{T} : \nabla \mathbf{v}_\mathcal{T} dx - \int_\Omega p_\mathcal{T} \operatorname{div} \mathbf{v}_\mathcal{T} + \int_\Omega Re[(\mathbf{u}_\mathcal{T} \cdot \nabla)\mathbf{u}_\mathcal{T}] \cdot \mathbf{v}_\mathcal{T}$$
$$+ \sum_{K \in \mathcal{T}} \delta_K h_K^2 \int_K Re[-\mathbf{f} - \Delta \mathbf{u}_\mathcal{T} + \nabla p_\mathcal{T} + Re(\mathbf{u}_\mathcal{T} \cdot \nabla)\mathbf{u}_\mathcal{T}] \cdot$$
$$\cdot [(\mathbf{u}_\mathcal{T} \cdot \nabla)\mathbf{v}_\mathcal{T}]$$
$$+ \sum_{K \in \mathcal{T}} \alpha_K \delta_K \int_K \operatorname{div} \mathbf{u}_\mathcal{T} \operatorname{div} \mathbf{v}_\mathcal{T} = \int_\Omega \mathbf{f} \cdot \mathbf{v}_\mathcal{T}$$
$$\int_\Omega q_\mathcal{T} \operatorname{div} \mathbf{u}_\mathcal{T} + \sum_{E \in \mathcal{E}} \delta_E h_E \int_E [p_\mathcal{T}]_E [q_\mathcal{T}]_E$$
$$+ \sum_{K \in \mathcal{T}} \delta_K h_K^2 \int_K [-\mathbf{f} - \Delta \mathbf{u}_\mathcal{T} + \nabla p_\mathcal{T} + Re(\mathbf{u}_\mathcal{T} \cdot \nabla)\mathbf{u}_\mathcal{T}] \cdot \nabla q_\mathcal{T} = 0$$

Properties of the Streamline Diffusion Method

- It is able to simultaneously stabilize the effects of the convection and of the divergence constraint.
- It gives rise to a differentiable discrete problem.
- Up to more technical arguments, its error analysis proceeds along the lines indicated before.
- It yields the same error estimates as before without the stability condition for the finite element spaces.
- In a mesh-dependent norm, it in addition gives control on $(\mathbf{u}_{Re} \cdot \nabla)(\mathbf{u}_{Re} - \mathbf{u}_{\mathcal{T},Re})$, the convective derivative of the error.

Potential Algorithms I

- Fixed-point iteration:
  - Requires the solution of Stokes problems.
  - Converges for sufficiently small Re.
- Newton iteration:
  - Requires the solution of linear Oseen problems with potentially large convection.
  - Converges quadratically if the starting value is sufficiently close to the solution.
  - May be combined with path-tracking.
- Path tracking:
  - Requires the solution of linear Oseen problems with potentially large convection.
  - May yield reasonable starting values for the Newton iteration.

Potential Algorithms II

- Non-linear CG-algorithm of Polak-Ribière:
  - Minimizes $\frac{1}{2}|\mathbf{u} - T(\mathbf{f} - Re(\mathbf{u} \cdot \nabla)\mathbf{u})|_1^2$.
  - Requires the solution of Stokes problems.
- Operator splitting:
  - Decouples the non-linearity and the incompressibility.
  - Requires the solution of Stokes problems and of non-linear Poisson equations for the components of the velocity.
- Multigrid algorithms:
  - May either be applied to the linear problems in an inner iteration or be used as an outer iteration with one of the above methods as smoothing method.

Fixed-Point Iteration

For $i = 0, 1, \ldots$ do:

- Solve the Stokes equations

$$\begin{aligned} -\Delta \mathbf{u}^{i+1} + \nabla p^{i+1} &= \{\mathbf{f} - Re(\mathbf{u}^i \cdot \nabla)\mathbf{u}^i\} && \text{in } \Omega \\ \operatorname{div} \mathbf{u}^{i+1} &= 0 && \text{in } \Omega \\ \mathbf{u}^{i+1} &= 0 && \text{on } \Gamma. \end{aligned}$$

- If $|\mathbf{u}^{i+1} - \mathbf{u}^i|_1 \le \varepsilon$ return $\mathbf{u}^{i+1}$, p^{i+1} as approximate solution; stop.

Newton Iteration

For $i = 0, 1, \ldots$ do:

- Solve the Oseen equations

$$\begin{aligned} -\Delta \mathbf{u}^{i+1} + \nabla p^{i+1} + Re(\mathbf{u}^i \cdot \nabla)\mathbf{u}^{i+1} & \\ + Re(\mathbf{u}^{i+1} \cdot \nabla)\mathbf{u}^i &= \{\mathbf{f} + Re(\mathbf{u}^i \cdot \nabla)\mathbf{u}^i\} && \text{in } \Omega \\ \operatorname{div} \mathbf{u}^{i+1} &= 0 && \text{in } \Omega \\ \mathbf{u}^{i+1} &= 0 && \text{on } \Gamma. \end{aligned}$$

- If $|\mathbf{u}^{i+1} - \mathbf{u}^i|_1 \le \varepsilon$ return $\mathbf{u}^{i+1}$, p^{i+1} as approximate solution; stop.

Path Tracking

Given a Reynolds' number λ, an increment $\Delta\lambda > 0$ and an approximate solution $\mathbf{u}_\lambda$ for the Navier-Stokes equations with $Re = \lambda$.

- Solve the Oseen equations

$$\begin{aligned} -\Delta \mathbf{v}_\lambda + \nabla q_\lambda + \lambda(\mathbf{u}_\lambda \cdot \nabla)\mathbf{v}_\lambda & \\ + \lambda(\mathbf{v}_\lambda \cdot \nabla)\mathbf{u}_\lambda &= \{\mathbf{f} - \lambda(\mathbf{u}_\lambda \cdot \nabla)\mathbf{u}_\lambda\} && \text{in } \Omega \\ \operatorname{div} \mathbf{v}_\lambda &= 0 && \text{in } \Omega \\ \mathbf{v}_\lambda &= 0 && \text{on } \Gamma. \end{aligned}$$

- Return $\mathbf{u}_\lambda + \Delta\lambda \mathbf{v}_\lambda$ as approximate solution of the Navier-Stokes equations with $Re = \lambda + \Delta\lambda$.

Operator Splitting

For $i = 0, 1, \ldots$ do:

- Solve the Stokes equations with no-slip boundary condition

$$2\omega \mathbf{u}^{i+\frac{1}{4}} - \Delta \mathbf{u}^{i+\frac{1}{4}} + \nabla p^{i+\frac{1}{4}} = 2\omega \mathbf{u}^i + \mathbf{f} - Re(\mathbf{u}^i \cdot \nabla)\mathbf{u}^i$$
$$\operatorname{div} \mathbf{u}^{i+\frac{1}{4}} = 0.$$

- Solve the non-linear Poisson equations with homogeneous boundary condition

$$\omega \mathbf{u}^{i+\frac{3}{4}} - \Delta \mathbf{u}^{i+\frac{3}{4}} + Re(\mathbf{u}^{i+\frac{3}{4}} \cdot \nabla)\mathbf{u}^{i+\frac{3}{4}} = \omega \mathbf{u}^{i+\frac{1}{4}} + \mathbf{f} - \nabla p^{i+\frac{1}{4}}.$$

- Solve the Stokes equations with no-slip boundary condition

$$2\omega \mathbf{u}^{i+1} - \Delta \mathbf{u}^{i+1} + \nabla p^{i+1} = 2\omega \mathbf{u}^{i+\frac{3}{4}} + \mathbf{f} - Re(\mathbf{u}^{i+\frac{3}{4}} \cdot \nabla)\mathbf{u}^{i+\frac{3}{4}}$$
$$\operatorname{div} \mathbf{u}^{i+1} = 0.$$

- If $|\mathbf{u}^{i+1} - \mathbf{u}^i|_1 \le \varepsilon$ return $\mathbf{u}^{i+1}$, p^{i+1}; stop.

Basic Idea

- For regular branches of solutions, a quantitative form of the implicit function theorem implies that error and residual are equivalent, i.e. the norm of the error can be bounded from above and from below by constant multiples of the dual norm of the residual.
- The dual norm of the residual can be estimated as for linear problems by
  - either evaluating element-wise the residual with respect to the strong form of the differential equation and suitable inter-element jumps
  - or solving auxiliary local discrete linear problems.
- Limit and bifurcation points can be treated by suitably augmenting the residual.

Equivalence of Error and Residual

- Assume that:
  - $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ are Banach spaces.
  - $F \in C^1(X, Y^*)$
  - $F(\varphi_0) = 0$
  - $DF(\varphi_0)$ is an isomorphism of X onto Y^*.
  - $|||DF(\varphi) - DF(\psi)||| \le \gamma \|\varphi - \psi\|_X$ for all φ, ψ in a ball with centre φ_0 and radius R_0.
- Set $R = \min\{R_0, \gamma^{-1}|||DF(\varphi_0)^{-1}|||^{-1}, 2\gamma^{-1}|||DF(\varphi_0)|||\}$.
- Then, the following error estimates hold for all φ in a ball with centre φ_0 and radius R:

$$\frac{1}{2}|||DF(\varphi_0)|||^{-1}\|F(\varphi)\|_{Y^*} \le \|\varphi - \varphi_0\|_X \le 2|||DF(\varphi_0)^{-1}|||\|F(\varphi)\|_{Y^*}.$$

Proof of the Equivalence

- The upper bound for $\|\varphi - \varphi_0\|_X$ follows from

$$\begin{aligned} &\varphi - \varphi_0 \\ &= DF(\varphi_0)^{-1}\Big\{DF(\varphi_0)(\varphi - \varphi_0) + F(\varphi) - F(\varphi) + F(\varphi_0)\Big\} \\ &= DF(\varphi_0)^{-1}\Big\{F(\varphi) \\ &\qquad + \int_0^1 \big[DF(\varphi_0) - DF(\varphi_0 + t(\varphi - \varphi_0))\big](\varphi - \varphi_0)dt\Big\}. \end{aligned}$$

- The lower bound for $\|\varphi - \varphi_0\|_X$ follows from

$$\begin{aligned} F(\varphi) &= DF(\varphi_0)(\varphi - \varphi_0) \\ &\quad + \int_0^1 \big[DF(\varphi_0 + t(\varphi - \varphi_0)) - DF(\varphi_0)\big](\varphi - \varphi_0)dt. \end{aligned}$$

Residual A Posteriori Error Estimates

- Residual a posteriori error indicator:

$$\eta_{R,K} = \Big\{h_K^2\|\mathbf{f}_\mathcal{T} + \Delta\mathbf{u}_\mathcal{T} - Re\,(\mathbf{u}_\mathcal{T}\cdot\nabla)\mathbf{u}_\mathcal{T} - \nabla p_\mathcal{T}\|_K^2 + \|\operatorname{div}\mathbf{u}_\mathcal{T}\|_K^2 + \frac{1}{2}\sum_{E\in\mathcal{E}_{K,\Omega}} h_E\|\big[\mathbf{n}_E\cdot(\nabla\mathbf{u}_\mathcal{T} - p_\mathcal{T}I)\big]_E\|_E^2\Big\}^{\frac{1}{2}}$$

- Upper bound:

$$\big\{|\mathbf{u}-\mathbf{u}_\mathcal{T}|_1^2 + \|p-p_\mathcal{T}\|^2\big\}^{\frac{1}{2}} \le c^*\Big\{\sum_{K\in\mathcal{T}}\big(\eta_{R,K}^2 + h_K^2\|\mathbf{f}-\mathbf{f}_\mathcal{T}\|_K^2\big)\Big\}^{\frac{1}{2}}$$

- Lower bound:

$$\eta_{R,K} \le c_*\big\{|\mathbf{u}-\mathbf{u}_\mathcal{T}|_{1,\omega_K}^2 + \|p-p_\mathcal{T}\|_{\omega_K}^2 + h_K^2\|\mathbf{f}-\mathbf{f}_\mathcal{T}\|_{\omega_K}^2\big\}^{\frac{1}{2}}$$

Non-Stationary Incompressible Navier-Stokes Equations

- Variational Formulation
- Finite Element Discretization
- Solution of the Discrete Problems
- A Posteriori Error Estimation and Adaptivity

Strong Form

- Non-stationary incompressible Navier-Stokes equations in dimensionless form with no-slip boundary condition

$$\begin{aligned}\frac{\partial\mathbf{u}}{\partial t} - \Delta\mathbf{u} + Re\,(\mathbf{u}\cdot\nabla)\mathbf{u} - \operatorname{grad} p = \mathbf{f} &\quad \text{in } \Omega\times(0,T)\\ \operatorname{div}\mathbf{u} = 0 &\quad \text{in } \Omega\times(0,T)\\ \mathbf{u} = 0 &\quad \text{on } \Gamma\times(0,T)\\ \mathbf{u}(\cdot,0) = \mathbf{u}_0 &\quad \text{in } \Omega\end{aligned}$$

- We want to multiply the momentum equation with a suitable test function $\mathbf{v}$ and integrate over $\Omega\times(0,T)$.
- We need appropriate spaces of univariate functions with values in suitable Sobolev spaces.

Function Spaces for Parabolic Problems

- $(U,\|\cdot\|_U)$, $(W,\|\cdot\|_W)$ Banach spaces, $U\hookrightarrow W$, $1\le p\le\infty$
- $L^p(a,b;U)$ all measurable functions $u:(a,b)\to U$ such that $t\mapsto\|u(\cdot,t)\|_U$ is in $L^p((a,b))$

$$\|u\|_{L^p(a,b;U)} = \begin{cases}\Big\{\int_a^b \|u(\cdot,t)\|_U^p dt\Big\}^{\frac{1}{p}}, & \text{if } p<\infty\\ \operatorname*{ess.sup}_{t\in(a,b)}\|u(\cdot,t)\|_U, & \text{if } p=\infty\end{cases}$$

- $W^p(a,b;U,W) = \big\{u\in L^p(a,b;U) : \partial_t u\in L^p(a,b;W)\big\}$

$$\|u\|_{W^p(a,b;U,W)} = \begin{cases}\Big\{\int_a^b \|u(\cdot,t)\|_U^p dt + \int_a^b \|\frac{\partial u}{\partial t}(\cdot,t)\|_W^p dt\Big\}^{\frac{1}{p}}\\ \operatorname*{ess.sup}_{t\in(a,b)} \max\Big\{\|u(\cdot,t)\|_U\ ,\ \|\frac{\partial u}{\partial t}(\cdot,t)\|_W\Big\}\end{cases}$$

Properties of the Function Spaces

- $L^p(a,b;U)$, $W^p(a,b;U,W)$ are Banach spaces.
- $\frac{\partial u}{\partial t}$ is defined in the distributional sense.
- Functions in $W^p(a,b;U,W)$ have traces $u(\cdot,a)$, $u(\cdot,b)$ in W if $p > 1$.

Variational Form

Find $\mathbf{u} \in L^2(0,T;H_0^1(\Omega)^d)$, $p \in L^2(0,T;L_0^2(\Omega))$ such that for all $\mathbf{v} \in W^\infty(0,T;H_0^1(\Omega)^d, L^2(\Omega)^d)$, $q \in L^\infty(L_0^2(\Omega))$

$$\int_0^T \int_\Omega \Big\{ -\mathbf{u} \cdot \frac{\partial \mathbf{v}}{\partial t} + \nabla \mathbf{u} : \nabla \mathbf{v} + Re\,[(\mathbf{u}\cdot\nabla)\mathbf{u}]\cdot \mathbf{v} - p \operatorname{div} \mathbf{v} \Big\} = \int_0^T \int_\Omega \mathbf{f}\cdot\mathbf{v} + \int_\Omega \mathbf{u}_0 \cdot \mathbf{v}(\cdot,0)$$

$$\int_0^T \int_\Omega q \operatorname{div} \mathbf{u} = 0$$

Equivalent Variational Form

Find $\mathbf{u} \in L^2(0,T;V)$ such that for all $\mathbf{v} \in W^\infty(0,T;V,L^2(\Omega)^d)$

$$\int_0^T \int_\Omega \Big\{ -\mathbf{u} \cdot \frac{\partial \mathbf{v}}{\partial t} + \nabla \mathbf{u} : \nabla \mathbf{v} + Re\,[(\mathbf{u}\cdot\nabla)\mathbf{u}]\cdot \mathbf{v} = \int_0^T \int_\Omega \mathbf{f}\cdot\mathbf{v} + \int_\Omega \mathbf{u}_0 \cdot \mathbf{v}(\cdot,0)$$

Properties of the Variational Problem

- For every Reynolds' number Re, every right-hand side $\mathbf{f} \in L^2(0,T;V^*)$ and every initial value $\mathbf{u}_0 \in L^2(\Omega)^d$ with $\operatorname{div} \mathbf{u}_0 = 0$, there exists at least one solution. It satisfies $\frac{\partial \mathbf{u}}{\partial t} \in L^1(0,T;V^*)$.
- If $d = 2$, there exists at most one solution. It satisfies $\frac{\partial \mathbf{u}}{\partial t} \in L^2(0,T;V^*)$ and $\mathbf{u} \in C([0,T],L^2(\Omega)^2)$.
- If $d = 3$, every solution satisfies $\frac{\partial \mathbf{u}}{\partial t} \in L^{\frac{4}{3}}(0,T;V^*)$ and $\mathbf{u} \in L^{\frac{8}{3}}(0,T;L^4(\Omega)^3)$. There is at most one solution in $L^2(0,T;V) \cap L^\infty(0,T;L^2(\Omega)^3) \cap L^8(0,T;L^4(\Omega)^3)$. Every such solution satisfies $\mathbf{u} \in C([0,T],L^2(\Omega)^3)$.

Existence. 1^{st} Step

- V is separable, i.e. there is as sequence of nested finite dimensional subspaces V_m such that $\bigcup_m V_m$ is dense in V.
- Denote by $\mathbf{u}_{0,m}$ the L^2-projection of $\mathbf{u}_0$ onto V_m.
- Recall that $\left[H_0^1(\Omega)^d\right]^3 \ni (\mathbf{u}, \mathbf{v}, \mathbf{w}) \mapsto \int_\Omega \left[(\mathbf{u} \cdot \nabla)\mathbf{v}\right] \cdot \mathbf{w}$ is a continuous anti-symmetric trilinear form.

Existence. 2^{nd} Step

- The theorem of Picard-Lindelöf implies that for every m there is a maximal $t_m \in (0, T]$ and a $\mathbf{v}_m \in C^1([0, t_m], V_m)$ with $\mathbf{v}_m(\cdot, 0) = \mathbf{u}_{0,m}$ such that for all $\mathbf{w}_m \in V_m$

$$\int_\Omega \left\{ \frac{\partial \mathbf{v}_m}{\partial t} \cdot \mathbf{w}_m + \nabla \mathbf{v}_m : \nabla \mathbf{w}_m + Re\left[(\mathbf{v}_m \cdot \nabla)\mathbf{v}_m\right] \cdot \mathbf{w}_m \right\}$$
$$= \int_\Omega \mathbf{f} \cdot \mathbf{w}_m.$$

- Inserting $\mathbf{w}_m = \mathbf{v}_m$ yields

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{v}_m\|^2 + |\mathbf{v}_m|_1^2 = \int_\Omega \frac{\partial \mathbf{v}_m}{\partial t} \cdot \mathbf{v}_m + |\mathbf{v}_m|_1^2$$
$$= \int_\Omega \mathbf{f} \cdot \mathbf{v}_m \leq \frac{1}{2}|\mathbf{v}_m|_1^2 + \frac{1}{2}|\mathbf{f}|_{-1}^2.$$

Existence. 3^{rd} Step

- This implies $\limsup_{t \to t_m} \|\mathbf{v}_m(\cdot, t)\| < \infty$.
- Hence, $t_m = T$ and $(\mathbf{v}_m)$ is contained in a bounded subset of $L^\infty(0, T; L^2(\Omega)^d) \cap L^2(0, T; V)$.
- A compactness theorem implies that there is a $\mathbf{u} \in L^\infty(0, T; L^2(\Omega)^d) \cap L^2(0, T; V)$ such that $\mathbf{v}_m \to \mathbf{u}$
  - weak in $L^2(0, T; V)$,
  - weak$*$ in $L^\infty(0, T; L^2(\Omega)^d)$,
  - strong in $L^2(0, T; L^2(\Omega)^d)$.
- The convergence allows to take the limit in the defining equation for the $\mathbf{v}_m$.
- Since $\bigcup V_m$ is dense in V, this proves the existence.

Uniqueness. 1^{st} Step

- Define operators A, N on $L^\infty(0, T; L^2(\Omega)^d) \cap L^2(0, T; V)$ by

$$\langle A\mathbf{u}, \mathbf{v} \rangle = \int_\Omega \nabla \mathbf{u} : \nabla \mathbf{v}, \quad \langle N(\mathbf{u}), \mathbf{v} \rangle = \int_\Omega \left[(\mathbf{u} \cdot \nabla)\mathbf{u}\right] \cdot \mathbf{v}.$$

- Then $|N(\mathbf{u})|_{-1} \leq \|\mathbf{u}\|_{L^4(\Omega)}^2$.
- Recall that $\|\mathbf{u}\|_{L^4(\Omega)} \leq 2^{\frac{d-1}{4}} \|\mathbf{u}\|^{1-\frac{d}{4}} |\mathbf{u}|_1^{\frac{d}{4}}$.
- Every solution of the variational problem satisfies

$$\frac{\partial \mathbf{u}}{\partial t} + A\mathbf{u} + N(\mathbf{u}) = \mathbf{f} \quad \text{in } V^*.$$

Uniqueness. 2nd Step ($d = 2$)

- Since $|N(\mathbf{u})|_{-1} \le \sqrt{2}\|\mathbf{u}\||\mathbf{u}|_1$, this proves that $A\mathbf{u}$, $N(\mathbf{u})$, and $\frac{\partial \mathbf{u}}{\partial t}$ are all in $L^2(0,T;V^*)$.
- Embedding theorems yield the remaining regularity results.
- The difference $\mathbf{w} = \mathbf{u}_1 - \mathbf{u}_2$ of any two solutions satisfies

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{w}(\cdot,t)\|^2 + |\mathbf{w}|_1^2 = \langle N(\mathbf{u}_1),\mathbf{w}\rangle - \langle N(\mathbf{u}_2),\mathbf{w}\rangle$$

$$= \int_\Omega \big[(\mathbf{w}\cdot\nabla)\mathbf{u}_1\big]\cdot\mathbf{w} \le \sqrt{2}|\mathbf{w}|_1\|\mathbf{w}\||\mathbf{u}_1|_1$$

$$\Rightarrow \frac{d}{dt}\|\mathbf{w}(\cdot,t)\|^2 \le |\mathbf{u}_1(\cdot,t)|_1^2\|\mathbf{w}(\cdot,t)\|^2$$

$$\Rightarrow \frac{d}{dt}\Big\{\|\mathbf{w}(\cdot,t)\|^2 \exp\Big[-\int_0^t |\mathbf{u}_1(\cdot,s)|_1^2 ds\Big]\Big\} \le 0.$$

- Since $\mathbf{u}_1 \in L^2(0,T;V)$ and $\mathbf{w}(\cdot,0) = 0$, this proves $\mathbf{w} = 0$.

Uniqueness. 3rd Step ($d = 3$)

- Since now $|N(\mathbf{u})|_{-1} \le \|\mathbf{u}\|_{L^4(\Omega)}^2 \le 2\|\mathbf{u}\|^{\frac{1}{2}}|\mathbf{u}|_1^{\frac{3}{2}}$, we have $A\mathbf{u} \in L^2(0,T;V^*)$ and $N(\mathbf{u})$, $\frac{\partial \mathbf{u}}{\partial t}$ in $L^{\frac{4}{3}}(0,T;V^*)$.
- Embedding theorems yield the remaining regularity results.
- The difference $\mathbf{w} = \mathbf{u}_1 - \mathbf{u}_2$ of any two solutions now satisfies

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{w}(\cdot,t)\|^2 + |\mathbf{w}|_1^2 = \int_\Omega \big[(\mathbf{w}\cdot\nabla)\mathbf{u}_1\big]\cdot\mathbf{w}$$

$$= -\int_\Omega \big[(\mathbf{w}\cdot\nabla)\mathbf{w}\big]\cdot\mathbf{u}_1 \le 2|\mathbf{w}|_1^{\frac{7}{4}}\|\mathbf{w}\|^{\frac{1}{4}}\|\mathbf{u}_1\|_{L^4(\Omega)}$$

$$\Rightarrow \frac{d}{dt}\|\mathbf{w}(\cdot,t)\|^2 \le \frac{1}{2}\big(\frac{7}{4}\big)^7\|\mathbf{u}_1(\cdot,t)\|_{L^4(\Omega)}^8\|\mathbf{w}(\cdot,t)\|^2$$

- Since $\mathbf{u}_1 \in L^8(0,T;L^4(\Omega)^3)$ and $\mathbf{w}(\cdot,0) = 0$, this proves $\mathbf{w} = 0$.

Regularity of Solutions

- Assume that
  - $d = 2$,
  - $\Gamma \in C^2$,
  - $\mathbf{f} \in L^\infty(0,T;L^2(\Omega)^2)$, $\frac{\partial \mathbf{f}}{\partial t} \in L^2(0,T;V^*)$, $\operatorname{div}\mathbf{f} = 0$,
  - $\mathbf{u}_0 \in H^2(\Omega)^2 \cap V$,
- then the solution is in $L^\infty(0,T;H^2(\Omega)^2)$.
- Assume that
  - $d = 3$,
  - $\Gamma \in C^\infty$,
  - $\mathbf{f} \in L^\infty(0,T;L^2(\Omega)^3)$, $\frac{\partial \mathbf{f}}{\partial t} \in L^1(0,T;L^2(\Omega)^3)$, $\operatorname{div}\mathbf{f} = 0$,
  - $\mathbf{u}_0 \in H^2(\Omega)^3 \cap V$,
  - $\|\mathbf{f}(\cdot,0)\|$, $\|\mathbf{f}\|_{L^\infty(0,T;V^*)}$ and $\|\mathbf{u}_0\|_2$ are sufficiently small,
- then the there is a unique solution with $\mathbf{u} \in L^\infty(0,T;H^2(\Omega)^3)$, $\frac{\partial \mathbf{u}}{\partial t} \in L^2(0,T;V) \cap L^\infty(0,T;L^2(\Omega)^3)$.

Further Regularity of Solutions

- The above regularity results do not suffice to prove second order error estimates w.r.t. time.
- In order to obtain such estimates, the quantities

$$|\frac{\partial \mathbf{u}}{\partial t}(\cdot,t)|_1,\ \int_t^T \|\frac{\partial \mathbf{u}}{\partial t}(\cdot,s)\|_2^2 ds,\ \int_t^T \|\frac{\partial^2 \mathbf{u}}{\partial t^2}(\cdot,s)\|^2 ds$$

must remain bounded for $t \to 0$.
- If any of these quantities remains bounded, the following overdetermined Neumann problem admits a unique solution:

$$\begin{aligned} \Delta\varphi &= \operatorname{div}\big(\mathbf{f}(\cdot,0) - (\mathbf{u}_0\nabla)\mathbf{u}_0\big) && \text{in } \Omega \\ \nabla\varphi &= \Delta\mathbf{u}_0 + \mathbf{f}(\cdot,0) - (\mathbf{u}_0\nabla)\mathbf{u}_0 && \text{on } \Gamma. \end{aligned}$$

- This is a non-local compatibility condition.

Discretization of Parabolic Problems

- There are three main approaches for the discretization of parabolic problems:
  - Method of lines,
  - Rothe's method,
  - Space-Time Finite Elements.
- For classical non-adaptive discretizations all approaches often yield the same discrete solution.
- The method of lines is very inflexible w.r.t. to adaptivity.
- The error analysis of Rothe's method is very intricate since it requires regularity results w.r.t. time which often are not available.
- Space-time finite elements are well amenable to a posteriori error estimation and space and time adaptivity.

Basic Idea of the Method of Lines

- Choose a fixed spatial mesh and associated finite element spaces.
- Apply a standard ODE-solver (e.g. implicit Euler, Crank-Nicolson, Runge-Kutta, ...) to the resulting system of ordinary differential equations.

Discretization of the Navier-Stokes Equations with the Method of Lines

- Choose a spatial mesh $\mathcal{T}$ and associated finite element spaces $X(\mathcal{T})$, $Y(\mathcal{T})$ which are uniformly stable for the Stokes problem.
- Denote by $A_{\mathcal{T}}$, $B_{\mathcal{T}}$ and $N_{\mathcal{T}}(\mathbf{u}_{\mathcal{T}})$ the associated stiffness matrices.
- Then the spatial discretization yields the following system of differential-algebraic equations:

$$\begin{aligned} \frac{d\mathbf{u}_{\mathcal{T}}}{dt} &= \mathbf{f}_{\mathcal{T}} - \nu A_{\mathcal{T}}\mathbf{u}_{\mathcal{T}} - B_{\mathcal{T}}p_{\mathcal{T}} - N_{\mathcal{T}}(\mathbf{u}_{\mathcal{T}}) \\ B_{\mathcal{T}}^T\mathbf{u}_{\mathcal{T}} &= 0. \end{aligned}$$

Temporal Discretization with the θ-Scheme

- Applying the θ-scheme to the above system requires to compute an appropriate interpolate $\mathbf{u}_{\mathcal{T}}^0 = R_{\mathcal{T}}\mathbf{u}_0$ of the initial value and, for $n = 1, 2, \ldots$, to solve the discrete stationary Navier-Stokes problems

$$\begin{aligned} \frac{\mathbf{u}_{\mathcal{T}}^n - \mathbf{u}_{\mathcal{T}}^{n-1}}{\tau_n} &= -B_{\mathcal{T}}p_{\mathcal{T}}^n + \theta\left\{\mathbf{f}_{\mathcal{T}}^n - \nu A_{\mathcal{T}}\mathbf{u}_{\mathcal{T}}^n - N_{\mathcal{T}}(\mathbf{u}_{\mathcal{T}}^n)\right\} \\ &\quad + (1-\theta)\left\{\mathbf{f}_{\mathcal{T}}^{n-1} - \nu A_{\mathcal{T}}\mathbf{u}_{\mathcal{T}}^{n-1} - N_{\mathcal{T}}(\mathbf{u}_{\mathcal{T}}^{n-1})\right\} \\ B_{\mathcal{T}}^T\mathbf{u}_{\mathcal{T}}^n &= 0 \end{aligned}$$

- The choice $\theta = \frac{1}{2}$ corresponds to the Crank-Nicolson scheme, $\theta = 1$ to the implicit Euler scheme.

Basic Idea of Rothe's Method

- Interpret the parabolic problem as an ordinary differential equation in a suitable infinite dimensional Banach space and apply a standard ODE-solver (e.g. implicit Euler, Crank-Nicolson, Runge-Kutta, ...).
- Every time-step then requires the solution of a stationary elliptic equation which is achieved by applying a standard finite element discretization.

Rothe's Method for the Navier-Stokes Equations

- Rothe's method in form of the θ-scheme requires to solve the following stationary non-linear elliptic equations for $n = 1, 2, \ldots$

$$\begin{aligned}
\frac{1}{\tau_n}\mathbf{u}^n - \theta\Delta\mathbf{u}^n + \theta Re\,(\mathbf{u}^n\cdot\nabla)\mathbf{u}^n - \operatorname{grad} p^n &\\
= \theta\mathbf{f}(\cdot,t_n) + \frac{1}{\tau_n}\mathbf{u}^{n-1} + (1-\theta)\{\mathbf{f}(\cdot,t_{n-1}) - \Delta\mathbf{u}^{n-1} &\\
+ Re\,(\mathbf{u}^{n-1}\cdot\nabla)\mathbf{u}^{n-1}\} & \quad \text{in } \Omega \\
\operatorname{div}\mathbf{u}^n = 0 & \quad \text{in } \Omega \\
\mathbf{u}^n = 0 & \quad \text{on } \Gamma.
\end{aligned}$$

- $\theta = \frac{1}{2}$ corresponds to the Crank-Nicolson scheme, $\theta = 1$ to the implicit Euler scheme.

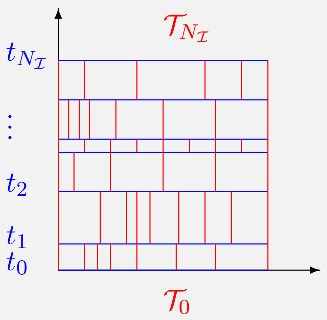
Spatial Discretization of the Stationary Problems

- The stationary problems only differ by the reaction term $\frac{1}{\tau_n}\mathbf{u}^n$ from the standard stationary Navier-Stokes equations.
- They can be discretized and solved as the standard Navier-Stokes equations.

Basic Idea of Space-Time Finite Element Methods

- Construct a space-time mesh for the space-time cylinder.
- In the variational formulation of the parabolic problem, replace the infinite dimensional spaces by finite dimensional approximations which consist of piece-wise polynomial functions w.r.t. time with values in spatial finite element spaces.

Space-Time Meshes



- Choose a partition $\mathcal{I} = \{[t_{n-1}, t_n] : 1 \le n \le N_\mathcal{I}\}$ of the time-interval $[0, T]$ with $0 = t_0 < \ldots < t_{N_\mathcal{I}} = T$.
- Set $\tau_n = t_n - t_{n-1}$.
- With every t_n associate an admissible, affine equivalent, shape regular partition $\mathcal{T}_n$ of Ω and finite element spaces $X_n = X(\mathcal{T}_n)$, $Y_n = Y(\mathcal{T}_n)$ for the velocity and pressure.

Conditions

- Non-Degeneracy: $\tau_n > 0$ for all n and $\mathcal{I}$.
- Transition Condition: For every n there is an affine equivalent, admissible, and shape-regular partition $\widetilde{\mathcal{T}}_n$ such that it is a refinement of both $\mathcal{T}_n$ and $\mathcal{T}_{n-1}$ and such that

$$\sup_{1\le n\le N_\mathcal{I}} \sup_{K\in\widetilde{\mathcal{T}}_n} \sup_{\substack{K'\in\mathcal{T}_n \\ K\subset K'}} \frac{h_{K'}}{h_K} < \infty$$

uniformly with respect to all partitions $\mathcal{I}$.

- Degree Condition: The polynomial degrees of the functions in X_n, Y_n are bounded uniformly w.r.t. all partitions $\mathcal{T}_n$ and $\mathcal{I}$.

Effect of the Transition Condition

- It restricts mesh-coarsening: It must not be too abrupt nor too strong.
- The method of characteristics below additionally requires a transition condition with reversed roles of $\mathcal{T}_{n-1}$ and $\mathcal{T}_n$.
- This restricts mesh-refinement: It must not be too abrupt nor too strong.
- Both restrictions are satisfied by the refinement and coarsening methods used in practice.

Space-Time Finite Element Discretization

Set $\mathbf{u}^0_{\mathcal{T}_0} = R_{\mathcal{T}_0}\mathbf{u}_0$ and successively determine $\mathbf{u}^n_{\mathcal{T}_n} \in X_n$, $p^n_{\mathcal{T}_n} \in Y_n$ such that for all $\mathbf{v}^n_{\mathcal{T}_n} \in X_n$, $q^n_{\mathcal{T}_n} \in Y_n$

$$\begin{aligned}
&\frac{1}{\tau_n}\int_\Omega \mathbf{u}^n_{\mathcal{T}_n}\cdot\mathbf{v}^n_{\mathcal{T}_n} + \theta\int_\Omega \nabla\mathbf{u}^n_{\mathcal{T}_n} : \nabla\mathbf{v}^n_{\mathcal{T}_n} - \int_\Omega p^n_{\mathcal{T}_n}\operatorname{div}\mathbf{v}^n_{\mathcal{T}_n} \\
&\quad + \Theta Re\int_\Omega [(\mathbf{u}^n_{\mathcal{T}_n}\cdot\nabla)\mathbf{u}^n_{\mathcal{T}_n}]\cdot\mathbf{v}^n_{\mathcal{T}_n} \\
&= \frac{1}{\tau_n}\int_\Omega \mathbf{u}^{n-1}_{\mathcal{T}_{n-1}}\cdot\mathbf{v}^n_{\mathcal{T}_n} + \theta\int_\Omega \mathbf{f}(\cdot,t_n)\cdot\mathbf{v}^n_{\mathcal{T}_n} \\
&\quad + (1-\theta)\int_\Omega \mathbf{f}(\cdot,t_{n-1})\cdot\mathbf{v}^n_{\mathcal{T}_n} + (1-\theta)\int_\Omega \nabla\mathbf{u}^{n-1}_{\mathcal{T}_{n-1}} : \nabla\mathbf{v}^n_{\mathcal{T}_n} \\
&\quad + (1-\Theta)Re\int_\Omega [(\mathbf{u}^{n-1}_{\mathcal{T}_{n-1}}\cdot\nabla)\mathbf{u}^{n-1}_{\mathcal{T}_{n-1}}]\cdot\mathbf{v}^n_{\mathcal{T}_n}
\end{aligned}$$

$$0 = \int_\Omega q^n_{\mathcal{T}_n}\operatorname{div}\mathbf{u}^n_{\mathcal{T}_n}$$

Choice of Parameters

- $\theta = \frac{1}{2}$ corresponds to the Crank-Nicolson scheme.
- $\theta = 1$ corresponds to the implicit Euler scheme.
- Due to the poor regularity for $t \to 0$, one usually uses the implicit Euler scheme for the first few time-steps.
- $\Theta = 1$ corresponds to a fully implicit treatment of the non-linear term. This requires the solution of discrete stationary Navier-Stokes equations in each time-step.
- $\Theta = 0$ corresponds to a fully explicit treatment of the non-linear term. This requires the solution of discrete Stokes equations in each time-step. As a compensation the size of the time-steps must be reduced drastically.
- The divergence constraint and the non-linear term may be stabilized as for stationary problems.

Overview

- All discretizations considered so far require the solution of a sequence of discrete stationary Navier-Stokes equations.
- At the expense of a drastically reduced time-step, the non-linear problems can be replaced by discrete Stokes problems.
- The method of characteristics, alias transport-diffusion algorithm, is particularly suited for the discretization of parabolic problems with a large convection term.
- It decouples the discretization of the temporal derivative and of the convection from the discretization of the diffusion terms.
- It requires the solution of a sequence of ODEs and of linear elliptic problems.

Basic Idea of the Method of Characteristics

- For every $\mathbf{v} \in V$ and every $(x^*, t^*) \in \Omega \times (0, T]$ the following characteristic equation admits a maximal solution which exists for all $t \in (0, t^*)$

$$\frac{d}{dt} x(t; x^*, t^*) = Re\, \mathbf{v}(x(t; x^*, t^*), t), \quad x(t^*; x^*, t^*) = x^*.$$

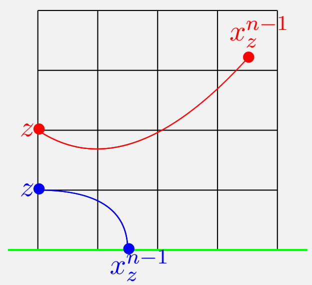
- $\mathbf{U}(x^*, t) = \mathbf{u}(x(t; x^*, t^*), t)$ satisfies

$$\frac{d\mathbf{U}}{dt} = \frac{\partial \mathbf{u}}{\partial t} + Re\, (\mathbf{v} \cdot \nabla)\mathbf{u}.$$

- The momentum equation therefore takes the form

$$\frac{d\mathbf{U}}{dt} - \Delta \mathbf{u} + \operatorname{grad} p = \mathbf{f}.$$

Re-Interpolation



- Assume that every function in X_n is determined by its values at a set $\mathcal{V}_n$ of nodes (Lagrange condition).
- For every n and $z \in \mathcal{V}_{n,\Omega}$ apply a classical ODE-solver to the characteristic equation associated with $(x^*, t^*) = (z, t_n)$ and denote by x_z^{n-1} the resulting approximation for $x(t_{n-1}; z, t_n)$.

The Method of Characteristics

- Determine $\widetilde{\mathbf{u}}_{\mathcal{T}_n}^{n-1} \in X_n$ such that $\widetilde{\mathbf{u}}_{\mathcal{T}_n}^{n-1}(z) = \mathbf{u}_{\mathcal{T}_{n-1}}^{n-1}(x_z^{n-1})$ for all $z \in \mathcal{V}_{n,\Omega}$.
- Find $\mathbf{u}_{\mathcal{T}_n}^n \in X_n$, $p_{\mathcal{T}_n}^n \in Y_n$ such that for all $\mathbf{v}_{\mathcal{T}_n}^n \in X_n$, $q_{\mathcal{T}_n}^n \in Y_n$

$$\frac{1}{\tau_n}\int_\Omega \mathbf{u}_{\mathcal{T}_n}^n \cdot \mathbf{v}_{\mathcal{T}_n}^n + \int_\Omega \nabla \mathbf{u}_{\mathcal{T}_n}^n : \nabla \mathbf{v}_{\mathcal{T}_n}^n - \int_\Omega p_{\mathcal{T}_n}^n \operatorname{div} \mathbf{v}_{\mathcal{T}_n}^n = \frac{1}{\tau_n}\int_\Omega \widetilde{\mathbf{u}}_{\mathcal{T}_n}^{n-1} \cdot \mathbf{v}_{\mathcal{T}_n}^n + \int_\Omega \mathbf{f}(\cdot, t_n) \cdot \mathbf{v}_{\mathcal{T}_n}^n$$

$$\int_\Omega q_{\mathcal{T}_n}^n \operatorname{div} \mathbf{u}_{\mathcal{T}_n}^n = 0$$

Properties

- Every time-step requires the solution of
  - an ODE for every node associated with X_n,
  - a discrete Stokes problem.
- The Stokes problems can be stabilized in the usual way.

Basic Steps of A Posteriori Error Estimation

- Use a parabolic energy estimate to prove that a suitable norm of the error can be bounded from above and from below by constant multiples of the corresponding dual norm of the residual.
- Appropriately split the residual in a contribution associated with the corresponding stationary problem and a complement which is associated with the temporal discretization.
- Prove that the norm of the residual can be bounded from above and from below by the sum of the norms of the two contributions.
- Bound the norms of the contributions separately using standard elliptic techniques for the part corresponding to the stationary problem.

Difficulty with the Navier-Stokes Equations

- There is no appropriate parabolic energy estimate available.
- This is due to the fact that the non-linear convection is too “strong” compared with the linear diffusion.
- This is reflected by the unsatisfactory regularity and uniqueness results for the the Navier-Stokes equations.
- For the non-stationary Stokes equations a suitable energy estimate is available. Then the main (technical) difficulty lies in the non-conformity $V(\mathcal{T}) \not\subset V$.

A Residual Error Indicator for the Non-Stationary Stokes Equations

- Spatial error indicator

$$\eta_h^n = \Big\{ \sum_{K\in\mathcal{T}_n} h_K^2 \|\mathbf{f}(\cdot,t_n) - \frac{1}{\tau_n}(\mathbf{u}_{\mathcal{T}_n}^n - \mathbf{u}_{\mathcal{T}_{n-1}}^{n-1}) + \Delta \mathbf{u}_{\mathcal{T}_n}^n - \nabla p_{\mathcal{T}_n}^n \|_K^2 + \sum_{E\in\mathcal{E}_{n,\Omega}} h_E \| [\mathbf{n}_E \cdot (\nabla \mathbf{u}_{\mathcal{T}_n}^n - p_{\mathcal{T}_n}^n \mathbf{I})]_E \|_E^2 + \sum_{K\in\mathcal{T}_n} \| \operatorname{div} \mathbf{u}_{\mathcal{T}_n}^n \|_K^2 \Big\}^{\frac{1}{2}}$$

- Temporal error indicator

$$\eta_\tau^n = \Big\{ \|\nabla(\mathbf{u}_{\mathcal{T}_n}^n - \mathbf{u}_{\mathcal{T}_{n-1}}^{n-1})\|^2 + \|\operatorname{div}(\mathbf{u}_{\mathcal{T}_n}^n - \mathbf{u}_{\mathcal{T}_{n-1}}^{n-1})\|^2 \Big\}^{\frac{1}{2}}$$

A Posteriori Error Estimates for the Non-Stationary Stokes Equations

- Denote by $\mathbf{u}_\mathcal{I}$ the continuous, temporally piece-wise affine function which coincides with $\mathbf{u}_{\mathcal{T}_n}^n$ at time t_n and by $p_\mathcal{I}$ the temporally piece-wise constant function which coincides with $p_{\mathcal{T}_n}^n$ on $(t_{n-1}, t_n]$.
- Then

$$\Big\{ \|\partial_t(\mathbf{u} - \mathbf{u}_\mathcal{I}) + \nabla(p - p_\mathcal{I})\|_{L^2(0,T;H^{-1}(\Omega)^d)}^2 + \|\mathbf{u} - \mathbf{u}_\mathcal{I}\|_{L^\infty(0,T;L^2(\Omega)^d)}^2 + \|\mathbf{u} - \mathbf{u}_\mathcal{I}\|_{L^2(0,T;H_0^1(\Omega)^d)}^2 \Big\}^{\frac{1}{2}}$$

$$\approx \Big\{ \|\mathbf{u}_{\mathcal{T}_0}^0 - \mathbf{u}_0\|^2 + \sum_{n=1}^{N_\mathcal{I}} \tau_n \big[(\eta_h^n)^2 + (\eta_\tau^n)^2 \big] \Big\}^{\frac{1}{2}}$$

Modifications for the Navier-Stokes Equations

- Add the non-linear term $Re\,(\mathbf{u}_{\mathcal{T}_n}^n \cdot \nabla)\mathbf{u}_{\mathcal{T}_n}^n$ to the element residuals in η_h^n.
- For every n determine $\widetilde{\mathbf{u}}_{\mathcal{T}_n}^n \in S_0^{1,0}(\mathcal{T}_n)^d$ such that for all $\mathbf{v}_{\mathcal{T}_n}^n \in S_0^{1,0}(\mathcal{T}_n)^d$

$$\int_\Omega \nabla \widetilde{\mathbf{u}}_{\mathcal{T}_n}^n : \nabla \mathbf{v}_{\mathcal{T}_n}^n = Re \int_\Omega \big[(\mathbf{u}_{\mathcal{T}_n}^n \cdot \nabla)\mathbf{u}_{\mathcal{T}_n}^n \big] \cdot \mathbf{v}_{\mathcal{T}_n}^n.$$

- Augment $(\eta_\tau^n)^2$ by

$$|\widetilde{\mathbf{u}}_{\mathcal{T}_n}^n|_1^2 + \sum_{K\in\mathcal{T}_n} h_K^2 \|\Delta \widetilde{\mathbf{u}}_{\mathcal{T}_n}^n + Re\,(\mathbf{u}_{\mathcal{T}_n}^n \cdot \nabla)(\mathbf{u}_{\mathcal{T}_n}^n - \mathbf{u}_{\mathcal{T}_{n-1}}^{n-1})\|_K^2.$$

An Algorithm for Space-Time Adaptivity

0. Given a tolerance ε, an initial mesh $\mathcal{T}_0$ and an initial time-step τ_1.

1. Refine $\mathcal{T}_0$ until $\|R_{\mathcal{T}_0}\mathbf{u}_0 - \mathbf{u}_0\| \le \frac{\varepsilon}{\sqrt{2}}$, set $n = 1$, $t_1 = \tau_1$.

2. Solve the discrete problem on time-level n and determine the error indicators η_h^n and η_τ^n.

3. If $\eta_\tau^n > \frac{\varepsilon}{2\sqrt{T}}$, replace t_n by $\frac{1}{2}(t_{n-1} + t_n)$ and return to **2**.

4. Apply a standard mesh-refinement and coarsening algorithm to the discrete problem on time-level n with the current time-step τ_n until $\eta_h^n \le \frac{\varepsilon}{2\sqrt{T}}$. If $\eta_\tau^n < \frac{\varepsilon}{4\sqrt{T}}$, replace τ_n by $2\tau_n$.

5. If $t_n = T$, stop. Otherwise set $t_{n+1} = \min\{T, t_n + \tau_n\}$, increment n by 1 and return to **2**.

Compressible and Inviscid Problems

- Systems in Divergence Form
- Discretization

The Setting

- Domain: $\Omega \subset \mathbb{R}^d$
- Source: $\mathbf{g} : \mathbb{R}^m \times \Omega \times (0, \infty) \to \mathbb{R}^m$
- Mass: $\mathbf{M} : \mathbb{R}^m \to \mathbb{R}^m$
- Flux: $\underline{\mathbf{F}} : \mathbb{R}^m \to \mathbb{R}^{m \times d}$
- Initial value: $\mathbf{U}_0 : \Omega \to \mathbb{R}^m$
- Problem: Find $\mathbf{U} : \Omega \times (0, \infty) \to \mathbb{R}^m$ such that under appropriate boundary conditions

$$\frac{\partial \mathbf{M}(\mathbf{U})}{\partial t} + \operatorname{div} \underline{\mathbf{F}}(\mathbf{U}) = \mathbf{g}(\mathbf{U}, x, t) \qquad \text{in } \Omega \times (0, \infty)$$
$$\mathbf{U}(\cdot, 0) = \mathbf{U}_0 \qquad \text{in } \Omega$$

- $\operatorname{div} \underline{\mathbf{F}}(\mathbf{U}) = \left(\sum_{j=1}^{d} \frac{\partial \underline{\mathbf{F}}(\mathbf{U})_{i,j}}{\partial x_j} \right)_{1 \le i \le m}$

Advective and Viscous Fluxes

- The flux $\underline{\mathbf{F}}$ splits into two contributions $\underline{\mathbf{F}} = \underline{\mathbf{F}}_{\text{adv}} + \underline{\mathbf{F}}_{\text{visc}}$.
- $\underline{\mathbf{F}}_{\text{adv}}$ is called advective flux and contains no derivatives.
- $\underline{\mathbf{F}}_{\text{visc}}$ is called viscous flux and contains spatial derivatives.
- The advective flux stems from the transport theorem and models transport or convection phenomena.
- The viscous flux models viscosity or diffusion phenomena.

Euler Equations

$$m = d + 2 \qquad \mathbf{U} = \begin{pmatrix} \rho \\ \mathbf{v} \\ e \end{pmatrix}$$

$$M(\mathbf{U}) = \begin{pmatrix} \rho \\ \rho \mathbf{v} \\ e \end{pmatrix} \qquad \mathbf{g} = \begin{pmatrix} 0 \\ \rho \mathbf{f} \\ \mathbf{f} \cdot \mathbf{v} \end{pmatrix}$$

$$\underline{\mathbf{F}}_{\text{adv}}(\mathbf{U}) = \begin{pmatrix} \rho \mathbf{v} \\ \rho \mathbf{v} \otimes \mathbf{v} + p \underline{\mathbf{I}} \\ e \mathbf{v} + p \mathbf{v} \end{pmatrix}$$

Compressible Navier-Stokes Equations in Conservative Form

$$m = d + 2 \qquad \mathbf{U} = \begin{pmatrix} \rho \\ \mathbf{v} \\ e \end{pmatrix}$$

$$M(\mathbf{U}) = \begin{pmatrix} \rho \\ \rho\mathbf{v} \\ e \end{pmatrix} \qquad \mathbf{g} = \begin{pmatrix} 0 \\ \rho\mathbf{f} \\ \mathbf{f}\cdot\mathbf{v} \end{pmatrix}$$

$$\underline{\mathbf{F}}_{\text{adv}}(\mathbf{U}) = \begin{pmatrix} \rho\mathbf{v} \\ \rho\mathbf{v}\otimes\mathbf{v} + p\underline{\mathbf{I}} \\ e\mathbf{v} + p\mathbf{v} \end{pmatrix} \qquad \underline{\mathbf{F}}_{\text{visc}}(\mathbf{U}) = \begin{pmatrix} 0 \\ \underline{\mathbf{T}} + p\underline{\mathbf{I}} \\ (\underline{\mathbf{T}} + p\underline{\mathbf{I}})\cdot\mathbf{v} + \sigma \end{pmatrix}$$

An Existence and Uniqueness Result

- Assume that
  - the boundary Γ is sufficiently smooth,
  - the exterior forces are sufficiently smooth,
  - the initial data ρ_0, $\mathbf{u}_0$, p_0 and the exterior forces satisfy appropriate compatibility conditions of the form

$$\begin{aligned} \mathbf{u}_0\cdot\mathbf{n} &= 0 \qquad \text{on } \Gamma \\ \frac{\partial\mathbf{u}_0}{\partial t}\cdot\mathbf{n} &= 0 \qquad \text{on } \Gamma \\ \Big(\frac{\partial\mathbf{f}}{\partial t} - (\frac{\partial\mathbf{u}_0}{\partial t}\cdot\nabla)\mathbf{u}_0 - (\mathbf{u}_0\cdot\nabla)\frac{\partial\mathbf{u}_0}{\partial t} & \\ +\nabla\big(\rho_0^{-1}p_0\,\text{div}(\rho_0\mathbf{u}_0)\big)\Big)\cdot\mathbf{n} &= 0 \qquad \text{on } \Gamma. \end{aligned}$$

- Then the compressible Navier-Stokes and Euler equations admit a unique solution on a small time interval.

Remarks

- The previous result is proved by a fixed-point argument.
- Long-time existence results require conditions of the form "initial data and exterior forces sufficiently small".
- Existence results under weaker assumptions can be proved using the concept of compensated compactness.
- Uniqueness results typically require some sort of entropy condition.

Most Popular Methods

- Finite Volume Methods

  Discretize the integral form of the system using piece-wise constant approximations on a mesh consisting of polyhedral cells combined with suitable numerical approximations for the fluxes across the cells' boundaries.
- Discontinuous Galerkin Methods

  Discretize a suitable weak formulation of the system using discontinuous piece-wise polynomial approximations combined with appropriate stabilization terms.

Finite Volume Discretization. 1st Step

- Choose a time-step $\tau > 0$.
- Choose a partition $\mathcal{T}$ of the domain Ω consisting of arbitrary non-overlapping polyhedra.
- Fix $n \in \mathbb{N}^*$ and $K \in \mathcal{T}$.
- Integrate the system over $K \times [(n-1)\tau, n\tau]$:

$$\int_{(n-1)\tau}^{n\tau} \int_K \frac{\partial \mathbf{M}(\mathbf{U})}{\partial t} + \int_{(n-1)\tau}^{n\tau} \int_K \operatorname{div} \underline{\mathbf{F}}(\mathbf{U}) = \int_{(n-1)\tau}^{n\tau} \int_K \mathbf{g}(\mathbf{U}, x, t)$$

Finite Volume Discretization. 2nd Step

Use integration by parts for the terms on the left-hand side:

$$\int_{(n-1)\tau}^{n\tau} \int_K \frac{\partial \mathbf{M}(\mathbf{U})}{\partial t} = \int_K \mathbf{M}(\mathbf{U}(x, n\tau)) - \int_K \mathbf{M}(\mathbf{U}(x, (n-1)\tau))$$

$$\int_{(n-1)\tau}^{n\tau} \int_K \operatorname{div} \underline{\mathbf{F}}(\mathbf{U}) = \int_{(n-1)\tau}^{n\tau} \int_{\partial K} \underline{\mathbf{F}}(\mathbf{U}) \cdot \mathbf{n}_K$$

Finite Volume Discretization. 3rd Step

- Assume that $\mathbf{U}$ is piecewise constant with respect to space and time.
- Denote by $\mathbf{U}_K^n$ and $\mathbf{U}_K^{n-1}$ its constant values on K at times $n\tau$ and $(n-1)\tau$ respectively:

$$\int_K \mathbf{M}(\mathbf{U}(x, n\tau)) \approx |K| \mathbf{M}(\mathbf{U}_K^n)$$

$$\int_K \mathbf{M}(\mathbf{U}(x, (n-1)\tau)) \approx |K| \mathbf{M}(\mathbf{U}_K^{n-1})$$

$$\int_{(n-1)\tau}^{n\tau} \int_{\partial K} \underline{\mathbf{F}}(\mathbf{U}) \cdot \mathbf{n}_K \approx \tau \int_{\partial K} \underline{\mathbf{F}}(\mathbf{U}_K^{n-1}) \cdot \mathbf{n}_K$$

$$\int_{(n-1)\tau}^{n\tau} \int_K \mathbf{g}(\mathbf{U}, x, t) \approx \tau |K| \mathbf{g}(\mathbf{U}_K^{n-1}, x_K, (n-1)\tau)$$

Finite Volume Discretization. 4th Step

Approximate the boundary integral for the flux by a numerical flux:

$$\tau \int_{\partial K} \underline{\mathbf{F}}(\mathbf{U}_K^{n-1}) \cdot \mathbf{n}_K \approx \tau \sum_{\substack{K' \in \mathcal{T} \\ \partial K \cap \partial K' \in \mathcal{E}}} |\partial K \cap \partial K'| \mathbf{F}_{\mathcal{T}}(\mathbf{U}_K^{n-1}, \mathbf{U}_{K'}^{n-1})$$

A Simple Finite Volume Scheme

- For every element $K \in \mathcal{T}$ compute

$$\mathbf{U}_K^0 = \frac{1}{|K|} \int_K \mathbf{U}_0(x).$$

- For $n = 1, 2, \ldots$ successively compute for all elements $K \in \mathcal{T}$

$$\begin{aligned}\mathbf{M}(\mathbf{U}_K^n) = \mathbf{M}(\mathbf{U}_K^{n-1}) \\ -\tau \sum_{\substack{K' \in \mathcal{T} \\ \partial K \cap \partial K' \in \mathcal{E}}} \frac{|\partial K \cap \partial K'|}{|K|} \mathbf{F}_{\mathcal{T}}(\mathbf{U}_K^{n-1}, \mathbf{U}_{K'}^{n-1}) \\ +\tau \mathbf{g}(\mathbf{U}_K^{n-1}, x_K, (n-1)\tau).\end{aligned}$$

Computational Fluid Dynamics
└Compressible and Inviscid Problems
└Discretization
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Possible Modifications

- The time-step may not be constant.
- The spatial mesh may vary with time.
- The approximation $\mathbf{U}_K^n$ may not be constant.

Computational Fluid Dynamics
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RUB

Open Tasks

- Construct the partition $\mathcal{T}$.
- Determine the numerical flux $\underline{\mathbf{F}}_{\mathcal{T}}$.
- Handle boundary conditions.

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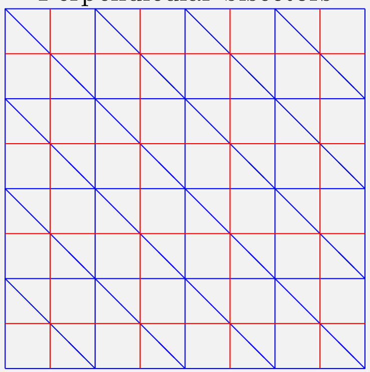
RUB

Dual Meshes

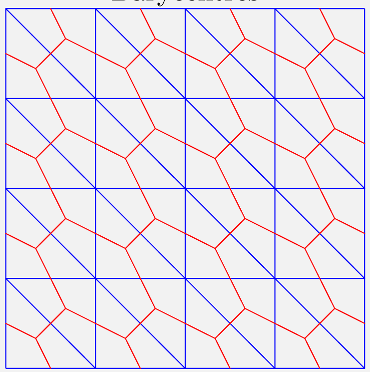
- Often the finite volume mesh $\mathcal{T}$ is constructed as a dual mesh departing from an admissible primal finite element mesh $\widetilde{\mathcal{T}}$.
- In two dimensions there are basically two approaches for the construction of dual meshes:
  - for every element $\widetilde{K} \in \widetilde{\mathcal{T}}$ draw the perpendicular bisectors at the midpoints of its edges,
  - for every element $\widetilde{K} \in \widetilde{\mathcal{T}}$ connect its barycentre with the midpoints of its edges.

Perpendicular Bisectors and Barycentres

Perpendicular bisectors

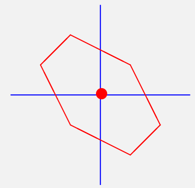


Barycentres

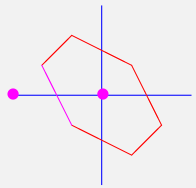


Properties of Dual Meshes

- Every element in $K \in \mathcal{T}$ can be associated with a vertex in x_K of $\widetilde{\mathcal{T}}$ and vice versa.



- With every edge E of $\mathcal{T}$ one may associate two vertices $x_{E,1}$, $x_{E,2}$ of $\widetilde{\mathcal{T}}$ such that the line $\overline{x_{E,1}\, x_{E,2}}$ intersects E.



Advantages and Disadvantages of Perpendicular Bisectors

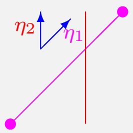
- The intersection of $\overline{x_{E,1}\, x_{E,2}}$ with E is perpendicular.
- The perpendicular bisectors of a triangle may intersect in a point outside the triangle. The intersection point is within the triangle only if its largest angle is at most a right one.
- The perpendicular bisectors of a quadrilateral may not intersect at all. They intersect in a common point inside the quadrilateral only if it is a rectangle.
- The construction with perpendicular bisectors has no three dimensional analogue.

Construction of Numerical Fluxes. Notations and Assumptions

- Assume that $\mathcal{T}$ is a dual mesh corresponding to a primal finite element mesh $\widetilde{\mathcal{T}}$.
- For every straight edge or face E of $\mathcal{T}$ denote by
  - K_1 and K_2 the adjacent volumes,
  - $\mathbf{U}_1$, $\mathbf{U}_2$ the values $\mathbf{U}_{K_1}^{n-1}$ and $\mathbf{U}_{K_2}^{n-1}$,
  - x_1, x_2 the vertices of $\widetilde{\mathcal{T}}$ such that the line $\overline{x_1\, x_2}$ intersects E.
- Split the numerical flux $\underline{\mathbf{F}}_{\mathcal{T}}(\mathbf{U}_1, \mathbf{U}_2)$ into a viscous numerical flux $\underline{\mathbf{F}}_{\mathcal{T},\mathrm{visc}}(\mathbf{U}_1, \mathbf{U}_2)$ and an advective numerical flux $\underline{\mathbf{F}}_{\mathcal{T},\mathrm{adv}}(\mathbf{U}_1, \mathbf{U}_2)$.

Approximation of Viscous Fluxes

- Introduce a local coordinate system $\eta_1, \ldots, \eta_d$ such that the direction η_1 is parallel to the direction $\overline{x_1\, x_2}$ and such that the other directions are tangential to E.



- Express all derivatives in $\underline{\mathbf{F}}_{\text{visc}}$ in terms of the new coordinate system.
- Suppress all derivatives not involving η_1.
- Approximate derivatives with respect to η_1 by difference quotients of the form $\frac{\varphi_1 - \varphi_2}{|x_1 - x_2|}$.

Spectral Decomposition of Advective Fluxes

- Denote by $C(\mathbf{V}) = D(\underline{\mathbf{F}}_{\text{adv}}(\mathbf{V}) \cdot \mathbf{n}_{K_1}) \in \mathbb{R}^{m \times m}$ the derivative of $\underline{\mathbf{F}}_{\text{adv}}(\mathbf{V}) \cdot \mathbf{n}_{K_1}$ with respect to $\mathbf{V}$.
- Assume that this matrix can be diagonalized (satisfied for Euler and compressible Navier-Stokes equations):

$$Q(\mathbf{V})^{-1} C(\mathbf{V}) Q(\mathbf{V}) = \Delta(\mathbf{V})$$

  with $Q(\mathbf{V}) \in \mathbb{R}^{m \times m}$ invertible and $\Delta(\mathbf{V}) \in \mathbb{R}^{m \times m}$ diagonal.
- Set $z^+ = \max\{z, 0\}$, $z^- = \min\{z, 0\}$ and

$$\Delta(\mathbf{V})^{\pm} = \operatorname{diag}\big(\Delta(\mathbf{V})_{11}^{\pm}, \ldots, \Delta(\mathbf{V})_{mm}^{\pm}\big),$$
$$C(\mathbf{V})^{\pm} = Q(\mathbf{V}) \Delta(\mathbf{V})^{\pm} Q(\mathbf{V})^{-1}.$$

Approximation of Advective Fluxes

- Steger-Warming

$$\mathbf{F}_{\mathcal{T},\text{adv}}(\mathbf{U}_1, \mathbf{U}_2) = C(\mathbf{U}_1)^+ \mathbf{U}_1 + C(\mathbf{U}_2)^- \mathbf{U}_2$$

- van Leer

$$\begin{aligned} &\mathbf{F}_{\mathcal{T},\text{adv}}(\mathbf{U}_1, \mathbf{U}_2) \\ &= \Big[C(\mathbf{U}_1) + C(\tfrac{1}{2}(\mathbf{U}_1 + \mathbf{U}_2))^+ - C(\tfrac{1}{2}(\mathbf{U}_1 + \mathbf{U}_2))^-\Big]\mathbf{U}_1 \\ &\quad + \Big[C(\mathbf{U}_2) - C(\tfrac{1}{2}(\mathbf{U}_1 + \mathbf{U}_2))^+ + C(\tfrac{1}{2}(\mathbf{U}_1 + \mathbf{U}_2))^-\Big]\mathbf{U}_2 \end{aligned}$$

Properties

- Both schemes require the computation of $D\underline{\mathbf{F}}_{\text{adv}}(\mathbf{V}) \cdot \mathbf{n}_{K_1}$ and of its eigenvalues and eigenvectors for suitable values of $\mathbf{V}$.
- The van Leer approximation in general is more costly than the Steger-Warming approximation since it requires three evaluations of $C(\mathbf{V})$ instead of two.
- For the compressible Navier-Stokes and Euler equations, however, this can be reduced to one evaluation since for these equations $\underline{\mathbf{F}}_{\text{adv}}(\mathbf{V}) \cdot \mathbf{n}_{K_1} = C(\mathbf{V})\mathbf{V}$ holds for all $\mathbf{V}$.

A One-Dimensional Example

- Burger's equation: $\dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} = 0$
- $F_{\text{adv}}(u) = \frac{1}{2}u^2$, $C(u) = u$, $C(u)^{\pm} = u^{\pm}$
- Steger-Warming:

$$F_{\mathcal{T},\text{adv}}(u_1,u_2) = \begin{cases} u_1^2 & \text{if } u_1 \ge 0, u_2 \ge 0 \\ u_1^2 + u_2^2 & \text{if } u_1 \ge 0, u_2 \le 0 \\ u_2^2 & \text{if } u_1 \le 0, u_2 \le 0 \\ 0 & \text{if } u_1 \le 0, u_2 \ge 0 \end{cases}$$

- van Leer:

$$F_{\mathcal{T},\text{adv}}(u_1,u_2) = \begin{cases} u_1^2 & \text{if } u_1 \ge -u_2 \\ u_2^2 & \text{if } u_1 \le -u_2 \end{cases}$$

TVD and ENO Schemes

- The convergence analysis of finite volume methods is based on compactness arguments, in particular the concept of compensated compactness.
- This requires to bound the total variation of the numerical approximation and to avoid unphysical oscillations.
- This leads to the concept of total variation diminishing TVD and essentially non-oscillating ENO schemes.
- Corresponding material may be found under the names of Enquvist, LeVeque, Osher, Roe, Tadmor, ...

Relation to Finite Element Methods

- Assume that $\mathcal{T}$ is a dual mesh corresponding to a primal finite element mesh $\widetilde{\mathcal{T}}$.
- There is a natural one-to-one correspondence between piece-wise constant functions on $\mathcal{T}$ and continuous piece-wise linear functions on $\widetilde{\mathcal{T}}$:

$$S^{0,-1}(\mathcal{T})^m \ni \mathbf{U}_{\mathcal{T}} \leftrightarrow \widetilde{\mathbf{U}}_{\widetilde{\mathcal{T}}} \in S^{1,0}(\widetilde{\mathcal{T}})^m$$
$$\mathbf{U}_{\mathcal{T}}|_K = \widetilde{\mathbf{U}}_{\widetilde{\mathcal{T}}}(x_K) \qquad \forall K \in \mathcal{T}.$$

A Simple Adaptive Algorithm

0. Given the solution $\mathbf{U}_{\mathcal{T}}$ of the finite volume scheme compute the corresponding finite element function $\widetilde{\mathbf{U}}_{\widetilde{\mathcal{T}}}$.
1. Apply a standard a posteriori error estimator to $\widetilde{\mathbf{U}}_{\widetilde{\mathcal{T}}}$.
2. Given the error estimator apply a standard mesh refinement and coarsening strategy to the finite element mesh $\widetilde{\mathcal{T}}$ and thus construct a new, locally refined and coarsened partition $\widehat{\mathcal{T}}$.
3. Use $\widehat{\mathcal{T}}$ to construct a new dual mesh $\mathcal{T}'$. This is the refinement of $\mathcal{T}$.

Idea of Discontinuous Galerkin Methods

- Approximate $\mathbf{U}$ by discontinuous functions which are polynomials w.r.t. space and time on small space-time cylinders of the form $K \times [(n-1)\tau, n\tau]$ with $K \in \mathcal{T}$.
- For every such cylinder multiply the differential equation by a corresponding test-polynomial and integrate the result over the cylinder.
- Use integration by parts for the flux term.
- Accumulate the contributions of all elements in $\mathcal{T}$.
- Compensate for the illegal partial integration by adding appropriate jump-terms across the element boundaries.
- Stabilize the scheme in a Petrov-Galerkin way by adding suitable element residuals.

A Simple Discontinuous Galerkin Scheme

- Compute $\mathbf{U}_{\mathcal{T}}^0$, the L^2-projection of $\mathbf{U}_0$ onto $S^{k,-1}(\mathcal{T})$.
- For $n \geq 1$ find $\mathbf{U}_{\mathcal{T}}^n \in S^{k,-1}(\mathcal{T})$ such that for all $\mathbf{V}_{\mathcal{T}}$

$$\begin{aligned}
&\sum_{K\in\mathcal{T}} \frac{1}{\tau} \int_K M(\mathbf{U}_{\mathcal{T}}^n) \cdot \mathbf{V}_{\mathcal{T}} - \sum_{K\in\mathcal{T}} \int_K \underline{\mathbf{F}}(\mathbf{U}_{\mathcal{T}}^n) : \nabla \mathbf{V}_{\mathcal{T}} \\
&+ \sum_{E\in\mathcal{E}} \delta_E h_E \int_E \left[\mathbf{n}_E \cdot \underline{\mathbf{F}}(\mathbf{U}_{\mathcal{T}}^n) \mathbf{V}_{\mathcal{T}}\right]_E \\
&+ \sum_{K\in\mathcal{T}} \delta_K h_K^2 \int_K \operatorname{div} \underline{\mathbf{F}}(\mathbf{U}_{\mathcal{T}}^n) \cdot \operatorname{div} \underline{\mathbf{F}}(\mathbf{V}_{\mathcal{T}}) \\
= &\sum_{K\in\mathcal{T}} \frac{1}{\tau} \int_K M(\mathbf{U}_{\mathcal{T}}^{n-1}) \cdot \mathbf{V}_{\mathcal{T}} + \sum_{K\in\mathcal{T}} \int_K \mathbf{g}(\cdot, n\tau) \cdot \mathbf{V}_{\mathcal{T}} \\
&+ \sum_{K\in\mathcal{T}} \delta_K h_K^2 \int_K \mathbf{g}(\cdot, n\tau) \cdot \operatorname{div} \underline{\mathbf{F}}(\mathbf{V}_{\mathcal{T}})
\end{aligned}$$

Possible Modifications

- The jump and stabilization terms can be chosen more judiciously.
- The time-step may not be constant.
- The spatial mesh may depend on time.
- The functions $\mathbf{U}_{\mathcal{T}}$ and $\mathbf{V}_{\mathcal{T}}$ may be piece-wise polynomials of higher order w.r.t. to time. Then the term $\sum_{K\in\mathcal{T}} \int_{(n-1)\tau}^{n\tau} \int_K \frac{\partial M(\mathbf{U}_{\mathcal{T}})}{\partial t} \cdot \mathbf{V}_{\mathcal{T}}$ must be added on the left-hand side and terms of the form $\frac{\partial M(\mathbf{U}_{\mathcal{T}})}{\partial t} \cdot \mathbf{V}_{\mathcal{T}}$ must be added to the element residuals.

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