# The delta invariant in Arakelov geometry

**Dissertation**

zur

Erlangung des Doktorgrades (Dr. rer. nat.)

der

Mathematisch-Naturwissenschaftlichen Fakultät

der

Rheinischen Friedrich-Wilhelms-Universität Bonn

vorgelegt von

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aus

Schalksmühle

Bonn, 2016

Angefertigt mit Genehmigung der Mathematisch-Naturwissenschaftlichen
Fakultät der Rheinischen Friedrich-Wilhelms-Universität Bonn

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Tag der Promotion: 21. April 2016

Erscheinungsjahr: 2016

## Abstract

In this thesis we study Faltings' delta invariant of compact and connected Riemann surfaces. This invariant plays a crucial role in Arakelov theory of arithmetic surfaces. For example, it appears in the arithmetic Noether formula. We give new explicit formulas for the delta invariant in terms of integrals of theta functions, and we deduce an explicit lower bound for it only in terms of the genus and an explicit upper bound for the Arakelov–Green function in terms of the delta invariant. Furthermore, we give a canonical extension of Faltings' delta invariant to the moduli space of indecomposable principally polarised complex abelian varieties. As applications to Arakelov theory, we obtain bounds for the Arakelov heights of the Weierstraß points and for the Arakelov intersection number of any geometric point with certain torsion line bundles in terms of the Faltings height. Moreover, we deduce an improved version of Szpiro's small points conjecture for cyclic covers of prime degree and an explicit expression for the Arakelov self-intersection number of the relative dualizing sheaf, an effective version of the Bogomolov conjecture and an arithmetic analogue of the Bogomolov–Miyaoka–Yau inequality for hyperelliptic curves.

# Contents

# Introduction

## From geometry to arithmetic

In number theory one is interested in so-called diophantine equations, that means integral (or rational) solutions of systems of equations

$$F_1(X_1,\dots,X_n)=0,\dots,F_m(X_1,\dots,X_n)=0,$$

where $F_j$ is a polynomial in the variables $X_1,\dots,X_n$. Equivalently, one can ask for integral (or rational) points on the algebraic variety defined by the polynomials $F_1,\dots,F_m$. A remarkable theorem according to this question is the Mordell conjecture, proved by Faltings in [Fal83]: Any smooth and projective curve $C$ of genus $g\geq 2$ defined over a number field $K$ has only finitely many $K$-rational points. However, it is still an open problem to find an effective bound for the heights of the $K$-rational points.

We can consider the analogous geometric situation. Let $k$ be an algebraically closed field of characteristic zero and $B'$ a smooth and projective curve of genus $q\geq 2$ defined over $k$. Denote by $k(B')$ the function field associated to $B'$. Let $C'$ be a smooth, projective, geometrically irreducible, nonconstant curve of genus $g\geq 2$ defined over $k(B')$. The minimal fibering $V\to B'$ associated to $C'$ is a complete algebraic surface. Hence, one has an intersection theory on $V$, and the finiteness of the $k(B')$-rational points on $C'$ follows from a suitable bound of the self-intersection number $K_V^2$ of the canonical bundle $K_V$ of $V$, as in [Par68, Theorem 5]. Here, one can also obtain an effective bound for the heights of the $k(B')$-rational points.

Coming back to the arithmetic situation of a smooth, projective and geometrically connected curve $C$ of genus $g\geq 2$ over a number field $K$, we obtain an arithmetic surface $p\colon \mathscr{C}\to B=\operatorname{Spec}\mathcal{O}_K$ by stable reduction theory, where $\mathcal{O}_K$ denotes the ring of integers of $K$. But this does not seem to be the counterpart of $V\to B'$ since $B$ is affine. In particular, it is not complete. Hence, we have no moving lemma to define an intersection theory. Thus, we should look for a model of $C$ over a compactification of $B$, but such a compactification does not exist in the category of schemes. Since the closed points of $B$ correspond to the non-archimedean valuations of $K$, we expect the missing points of $B$ to correspond to the archimedean valuations of $K$.

Instead of constructing a compactification of $B$, we enrich the data of any vector bundle $E$ over $\mathscr{C}$ by a structure corresponding to the archimedean valuations of $K$. If $P\colon B \to \mathscr{C}$ denotes a section of $p$ and $b \in |B|$ a non-archimedean valuation of $K$, the completed stalk $(P^*E)_b$ is a vector bundle on the completion Spec $\mathcal{O}_{K,b}$, that means a vector space $E_{K_b}$ over $K_b$, the completion of $K$ with respect to $b$, together with a $\mathcal{O}_{K,b}$-lattice $\Lambda \subseteq E_{K_b}$. Any $\mathcal{O}_{K,b}$-lattice $\Lambda \subseteq E_{K_b}$ defines a maximal compact subgroup in the $b$-adic topology $\mathrm{GL}_{\mathcal{O}_{K,b}}(\Lambda) = \{g \in \mathrm{GL}(E_{K_b}) | g(\Lambda) = \Lambda\} \subseteq \mathrm{GL}(E_{K_b})$. This correspondence defines a bijection between $\mathcal{O}_{K,b}$-lattices up to similarity in $E_{K_b}$ and maximal compact subgroups of $\mathrm{GL}(E_{K_b})$ in the $b$-adic topology, where two lattices $\Lambda$ and $\Lambda'$ are similar if there exists a number $c \in K_b \setminus \{0\}$ such that $c\Lambda = \Lambda'$.

Now let $v$ be any archimedean place of $K$. After a finite field extension, we may assume that the completion of $K$ with respect to $v$ is $\mathbb{C}$. Let $E_{\mathbb{C}}$ be the base change of $P^*E$ induced by an embedding $\mathcal{O}_K \to \mathbb{C}$ corresponding to $v$. The maximal compact subgroups of $\mathrm{GL}(E_{\mathbb{C}})$ bijectively correspond to the positive-definite hermitian forms up to multiples in $\mathbb{C}\setminus\{0\}$. Thence, we think of a vector bundle on a "*model of $C$ over the compactification of* Spec $\mathcal{O}_K$" as a vector bundle $E$ on $\mathscr{C}$ together with a smooth hermitian metric on $E_\sigma$, for every embedding $\sigma\colon \mathcal{O}_K \to \mathbb{C}$, compatible with the complex conjugation, where $E_\sigma$ denotes the base change of $E$ induced by $\sigma$. This motivates the idea of Arakelov geometry.

# Arakelov geometry

We review the main ideas of Arakelov geometry. The main references for this section are [Ara74] and [Fal84]. Arakelov introduced in [Ara74] a new kind of divisors, now called Arakelov divisors, and he defined an intersection theory for them. An Arakelov divisor $D$ can be written as a formal sum $D = D_{\mathrm{fin}} + \sum_{v\in S_\infty} r_v \cdot F_v$, where $D_{\mathrm{fin}}$ denotes a classical divisor on $\mathscr{C}$, the sum runs over all archimedean valuations $v$ of $K$, $r_v \in \mathbb{R}$ is any real number and $F_v$ is a formal symbol standing for the *"fibre above $v$"*. For example, the Arakelov divisor associated to a rational function $f \in K(C)$ is given by

$$\widehat{\mathrm{div}}(f) = \mathrm{div}(f) + \sum_{\sigma\colon K\to\mathbb{C}} \left( -\int_{C_\sigma} \log|f|_\sigma \mu \right) F_\sigma,$$

where $\mathrm{div}(f)$ is the usual Weil divisor associated to $f$, the sum runs over all embeddings $\sigma\colon K\to\mathbb{C}$, $C_\sigma$ is the pullback of $C$ induced by the embedding $\sigma\colon K\to\mathbb{C}$, we write $F_\sigma=F_v$ if $\sigma\colon K\to\mathbb{C}$ corresponds to the archimedean valuation $v$, and $\mu$ denotes the canonical Arakelov $(1,1)$-form given by $\frac{i}{2g}\sum_{j=1}^g\psi_j\wedge\overline{\psi_j}$ for a basis $\psi_1,\dots,\psi_g$ of $H^0(C_\sigma,\Omega^1_{C_\sigma})$, which is orthonormal with respect to the inner product $\langle\omega,\omega'\rangle=\frac{i}{2}\int\omega\wedge\overline{\omega'}$. We denote by $\widehat{\mathrm{Div}}(C)$ the group of Arakelov divisors on $C$ and by

$$\widehat{\mathrm{Ch}}(C)=\widehat{\mathrm{Div}}(C)/\{\widehat{\mathrm{div}}(f)|f\in K(C)\}$$

the Arakelov–Chow group.

Alternatively, we can consider metrized line bundles on $\mathscr{C}$, that means line bundles $L$ on $\mathscr{C}$ together with a hermitian metric on $L_\sigma$ for every embedding $\sigma\colon K\to\mathbb{C}$, compatible with complex conjugation. For a compact and connected Riemann surface $X$ of genus $g\geq 1$ the Arakelov–Green function $G\colon X^2\to\mathbb{R}_{\geq 0}$ is the unique function satisfying $\partial_P\overline{\partial}_P\log G(P,Q)=\pi i(\mu-\delta_Q)$ and $\int_X\log G(P,Q)\mu(P)=0$. For any section $Q$ of $p\colon\mathscr{C}\to B$ we get a canonical metric on the line bundle $\mathcal{O}_{\mathscr{C}}(Q)$ by putting $\|\mathbf{1}_Q\|(P)=G(P,Q)$ on the line bundle $\mathcal{O}_{C_\sigma}(Q)$ on $C_\sigma$ for every embedding $\sigma$, where $\mathbf{1}_Q\in H^0(\mathscr{C},\mathcal{O}_{\mathscr{C}}(Q))$ denotes the canonical constant section.

Let $D=D_{\mathrm{fin}}+\sum_{v\in S_\infty}r_v\cdot F_v$ be any Arakelov divisor. We also obtain a canonical metric on $\mathcal{O}(D_{\mathrm{fin}})$ by forcing that the metric is compatible with tensor products of line bundles and equipping bundles of the form $\mathcal{O}_{\mathscr{C}}(F_b)$ with the canonical metric, where $b$ is a closed point of $B$ and $F_b$ is the fibre of $p$ over $b$. We associate to $D$ the metrized line bundle $\mathcal{O}(D)$, where the underlying line bundle is $\mathcal{O}(D_{\mathrm{fin}})$ and its metric over $C_\sigma$ for any embedding $\sigma$ corresponding to an archimedean valuation $v$ is $e^{-r_v}$ times its canonical metric. We call a metrized line bundle $L$ admissible if there is an Arakelov divisor $D$ with $L\cong\mathcal{O}(D)$ as metrized line bundles. We denote by $\widehat{\mathrm{Pic}}(\mathscr{C})$ the group of isomorphism classes of admissible metrized line bundles on $\mathscr{C}$. We have a canonical isomorphism $\widehat{\mathrm{Ch}}(\mathscr{C})\cong\widehat{\mathrm{Pic}}(\mathscr{C})$. A metrized line bundle is admissible if and only if it holds $\partial\overline{\partial}\log\|s_\sigma\|^2_\sigma=2\pi i\deg(L_\sigma)\mu$ on $L_\sigma$ for every embedding $\sigma\colon K\to\mathbb{C}$ and for a local generating section $s_\sigma\in H^0(C_\sigma,L_\sigma)$.

Arakelov defined in [Ara74] a bilinear and symmetric intersection pairing on $\widehat{\mathrm{Pic}}(\mathscr{C})$. For example, for two sections $P,Q$ of $p$, which are different on the generic fibre, the Arakelov intersection number is defined by

$$(P,Q)=(\mathcal{O}(P),\mathcal{O}(Q))=\sum_{v\in|B|}(P|_{\mathscr{C}_{\overline{v}}},Q|_{\mathscr{C}_{\overline{v}}})\log N_v-\sum_{\sigma\colon K\to\mathbb{C}}\log G_\sigma(P,Q),$$

where $|B|$ is the set of closed points in $B$, $(P|_{\mathscr{C}_{\overline{v}}},Q|_{\mathscr{C}_{\overline{v}}})$ denotes the usual intersection number on the geometric fibre $\mathscr{C}_{\overline{v}}$ of $\mathscr{C}$ over $v$, $N_v$ denotes the cardinality of the residue field of $\mathcal{O}_K$ at $v$ and $G_\sigma$ is the Arakelov–Green function of $C_\sigma$.

Let $\omega_{\mathscr{C}/B}$ be the relative dualizing sheaf of $p\colon \mathscr{C} \to B$. There is a canonical metric on $(\omega_{\mathscr{C}/B})_\sigma$ induced by $\|dz\|_{\mathrm{Ar}}(P) = \lim_{Q\to P} |z(Q) - z(P)| / G_\sigma(P,Q)$, where $z\colon U \to \mathbb{C}$ is a local coordinate of a neighbourhood $P \in U \subseteq C_\sigma$. This metric is admissible, see [Ara74]. We also have a canonical metric on the line bundle $\det p_*\omega_{\mathscr{C}/B}$ induced by the inner product $\langle \omega, \omega' \rangle = \frac{i}{2}\int \omega \wedge \overline{\omega'}$ on $H^0(C_\sigma, \Omega^1_{C_\sigma})$. The Arakelov degree of a metrized line bundle $L$ on $B$ is defined by

$$\widehat{\deg} L = \sum_{v\in|B|} \mathrm{ord}_v(s) \log N_v - \sum_{\sigma\colon K\to\mathbb{C}} \log \|s\|_\sigma,$$

where $s \in H^0(B, L)$ is a non-zero section of $L$ and $\mathrm{ord}_v(s)$ denotes the order of vanishing of $s$ at $v$.

After a finite field extension, we can assume that $C$ has semi-stable reduction over $K$. Further, we denote by $\mathscr{C}$ the minimal regular model of $C$ over $B$. Then the arithmetic Noether formula, proved by Faltings [Fal84, Theorem 6], states

$$12\widehat{\deg} \det p_*\omega_{\mathscr{C}/B} = (\omega_{\mathscr{C}/B}, \omega_{\mathscr{C}/B}) + \sum_{v\in|B|} \delta_v \log N_v + \sum_{\sigma\colon K\to\mathbb{C}} \delta'(C_\sigma), \tag{1}$$

where $\delta_v$ denotes the number of singularities of the geometric fibre $\mathscr{C}_{\overline{v}}$ of $\mathscr{C}$ over $v \in |B|$ and $\delta'$ is a certain invariant of compact and connected Riemann surfaces of positive genus. This invariant is the main object of study in this thesis. However, we will consider the normalization $\delta(X) = \delta'(X) + 4g\log 2\pi$, as originally introduced by Faltings, see [Mor89] for the comparison of these normalizations.

# The delta invariant

Next, we discuss some properties of the invariant $\delta$. Faltings introduced the invariant $\delta$ in [Fal84] by the following equation. Let $X$ be any compact and connected Riemann surface of genus $g \geq 1$. Then $\delta(X)$ satisfies

$$\|\theta\|(P_1 + \cdots + P_g - Q) = \exp(-\tfrac{1}{8}\delta(X)) \cdot \frac{\|\det(\psi_j(P_k))\|_{\mathrm{Ar}}}{\prod_{j<k} G(P_j, P_k)} \cdot \prod_{j=1}^{g} G(P_j, Q), \tag{2}$$

where $P_1, \ldots, P_g, Q \in X$ are pairwise different points, such that the line bundle $\mathcal{O}(P_1 + \cdots + P_g - Q)$ has no global sections and $\|\theta\|\colon \mathrm{Pic}_{g-1}(X) \to \mathbb{R}_{\geq 0}$

is the unique function satisfying (i) $\partial\overline{\partial}\log\|\theta\| = \pi i(\nu - \delta_\Theta)$ for $\Theta \subseteq \mathrm{Pic}_{g-1}(X)$ the divisor consisting of line bundles of degree $(g-1)$ having global sections and $\nu$ the canonical translation invariant $(1,1)$-form with $\int_{\mathrm{Pic}_{g-1}(X)} \nu^g = g!$ and (ii) $\int_{\mathrm{Pic}_{g-1}(X)} \|\theta\|^2 \frac{\nu^g}{g!} = 2^{-g/2}$.

Since this formula is very implicit, it would be nice to have a more explicit description for $\delta$. For any complex elliptic curve $E$ Faltings proved

$$\delta(E) = -\log\|\Delta_1\|(E) - 8 \cdot \log(2\pi),$$

where we set

$$\|\Delta_1\|(E) = (\mathrm{Im}\ \tau)^6 \cdot \exp(-2\pi \mathrm{Im}\ \tau) \cdot \prod_{n=1}^{\infty} |1 - \exp(2\pi i n\tau)|^{24}$$

for $\tau \in \mathbb{C}$ satisfying $\mathrm{Im}\ \tau > 0$ and $E \cong \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$. A few years later, Bost stated in [Bos87, Proposition 4] the following explicit formula for any compact and connected Riemann surface $X$ of genus $g = 2$

$$\delta(X) = -4\int_{\mathrm{Pic}_1(X)} \log\|\theta\| \frac{\nu^2}{2} - \frac{1}{4}\log\|\Delta_2\|(X) - 16\log 2\pi, \qquad (3)$$

where we define for any hyperelliptic Riemann surface $X$

$$\|\Delta_g\|(X) = 2^{-4(g+1)\binom{2g}{g-1}} \cdot \prod_{\substack{L \in \mathrm{Pic}_{g-1}(X) \\ L^{\otimes 2} \cong K_X, \|\theta\|(L) \neq 0}} \|\theta\|(L)^8$$

with $K_X$ the canonical bundle on $X$. We will show that Bost's formula can be generalized to hyperelliptic Riemann surfaces of arbitrary genus.

Guàrdia [Guà99, Proposition 1.1] and de Jong [dJo08, Theorem 4.4] constructed derivative versions of the function $\|\theta\|$ to replace $\|\det(\psi_j(P_k))\|_{\mathrm{Ar}}$ in (2). Since $\|\theta\|$ can explicitly be given by the Riemann theta function associated to $X$, it remains to replace the Arakelov–Green function in (2). This was also done by de Jong using Weierstraß points in [dJo05a, Theorem 4.4]. In particular, he found the following formula for hyperelliptic Riemann surfaces

$$\delta(X) = \frac{4(g-1)}{g^2}\int_X \log\|\theta\|(gP - Q)\mu(P) - \frac{3g-1}{2g\binom{2g}{g-1}}\log\|\Delta_g\|(X) - 8g\log 2\pi, \qquad (4)$$

see [dJo05b, Corollary 1.7]. We will prove an expression for $\delta(X)$ only in terms of integrals of $\|\theta\|$ and one of its derivative versions.

The arithmetic Noether formula predicts that $\delta(X)$ is the archimedean counterpart of the number of singularities $\delta_v$ of a geometric fibre $\mathscr{C}_{\overline{v}}$. Hence, one expects that $\lim_{t\to 0}\delta(\mathscr{X}_t)$ becomes infinity for a smooth family of semi-stable complex curves $\mathscr{X}\to D$ over the unit disc $D\subseteq\mathbb{C}$, where $\mathscr{X}_t$ is a Riemann surface if and only if $t\neq 0$. Indeed, Jorgenson [Jor90] and Wentworth [Wen91] proved that if $\mathscr{X}_0$ consists of two Riemann surfaces of genus $g_1$ and $g_2$ meeting in one node, then $\delta(\mathscr{X}_t)+\frac{4g_1g_2}{g}\log|t|$ is bounded on $D$ and if $\mathscr{X}_0$ has only one non-separating node as a singular point, then $\delta(\mathscr{X}_t)+\frac{4g-1}{3g}\log|t|+6\log(-\log|t|)$ is bounded on $D$. In particular, $\delta$ becomes infinity on the boundary of $\mathcal{M}_g$, the moduli space of compact and connected Riemann surfaces of genus $g$, in its Deligne–Mumford compactification $\overline{\mathcal{M}}_g$. A more general result on the degeneration of $\delta(\mathscr{X}_t)$, where $\mathscr{X}_0$ is any semi-stable complex curve, was recently proved by de Jong [dJo15].

It follows, that $\delta$ is bounded from below on $\mathcal{M}_g$, and it is natural to ask for an effective lower bound. Van Känel deduced the lower bounds $\delta(X)\geq -9$ for $X$ of genus $g=1$, see [vKä14b, p. 92], and $\delta(X)\geq -186$ for $X$ of genus $g=2$, see [vKä14a, Proposition 5.1], from the explicit formulas above. Jorgenson and Kramer [JK09, JK14] also found effective lower and upper bounds of $\delta(X)$ in terms of hyperbolic geometry and Javanpeykar [Jav14] proved lower and upper bounds of $\delta(X)$ in terms of the Belyi degree of $X$. However, these bounds are not bounded from below on $\mathcal{M}_g$. In this thesis we will prove, that we have in general $\delta(X)>-2g\log 2\pi^4$.

As an arithmetic application, Parshin [Par90] proved, that if there are absolute constant $c_1$, $c_2$ and $c_3$, such that every curve $C/K$ of genus $g\geq 2$ satisfies

$$(\omega_{\mathscr{C}/B},\omega_{\mathscr{C}/B})\leq c_1\left(\sum_{v\in|B|}\delta_v\log N_v+\sum_{\sigma\colon K\to\mathbb{C}}\delta(C_\sigma)\right)+c_2\log D_K+c_3[K:\mathbb{Q}], \tag{5}$$

then one can deduce an effective version of the Mordell conjecture, and even the *abc*-conjecture. Here, $D_K$ denotes the absolute value of the discriminant of $K/\mathbb{Q}$ and the constants are allowed to depend on $g$. We will obtain such an inequality for hyperelliptic curves.

Zhang proved in [Zha93, Theorem 5.6] that a suitable lower bound of $(\omega_{\mathscr{C}/B},\omega_{\mathscr{C}/B})$ leads to an effective Bogomolov conjecture. The Bogomolov conjecture states, that any embedding of $C(\overline{K})$ into $\mathrm{Pic}_0(C)(\overline{K})$ is discrete with respect to the Neron-Tate norm, where $\overline{K}$ denotes an algebraic closure of $K$. It was non-effectively proved by Ullmo [Ull98] and Zhang [Zha98] in 1998. By the arithmetic Noether formula, an explicit description of $\widehat{\deg}\det p_*\omega_{\mathscr{C}/B}$

and $\delta(C_\sigma)$ give an explicit description of $(\omega_{\mathscr{C}/B}, \omega_{\mathscr{C}/B})$. We will deduce an effective Bogomolov conjecture for hyperelliptic curves in this way.

There are two other invariants of Riemann surfaces motivated by Arakelov theory. The Kawazumi–Zhang invariant $\varphi$, introduced independently in [Kaw08] and [Zha10], appears by comparing the self-intersection numbers $(\omega_{\mathscr{C}/B}, \omega_{\mathscr{C}/B})$ and $(\Delta_\xi, \Delta_\xi)$, where $\Delta_\xi$ denotes the canonical Gross–Schoen cycle, see [Zha10]. It holds

$$\tfrac{2g+1}{2g-2}(\omega_{\mathscr{C}/B}, \omega_{\mathscr{C}/B}) = (\Delta_\xi, \Delta_\xi) + \sum_{\sigma\colon K\to\mathbb{C}} \varphi(C_\sigma) + \sum_{v\in|B|} \left(\tfrac{2g+1}{2g-2}\epsilon_v + \varphi_v\right) \log N_v,$$

where $\epsilon_v$ and $\varphi_v$ are certain invariants of the weighted dual graph of the geometric fibre $\mathscr{C}_{\overline{v}}$, see also [Zha10] for their definitions. The Hain–Reed invariant $\beta_g$, introduced in [HR04], appears as a quotient of two canonical metrics on $(\det p_*\omega_{\mathscr{C}/B})^{\otimes 8g+4}$. De Jong [dJo13, Theorem 1.4] obtained the relation

$$\delta(X) = \tfrac{3}{2g+1}\beta_g(X) - \tfrac{2g-2}{2g+1}\varphi(X)$$

for any compact and connected Riemann surface $X$ of genus $g \geq 1$.

# Statement of results

In the following we summarize the results of this thesis.

## Results on Riemann surfaces

Our main result, stated in the following theorem, gives a relation between Faltings' $\delta$-invariant and the Kawazumi–Zhang invariant $\varphi$.

**Theorem 1.** *Any compact and connected Riemann surface $X$ of genus $g \geq 1$ satisfies*

$$\delta(X) = -24 \int_{\mathrm{Pic}_{g-1}(X)} \log \|\theta\| \frac{\nu^g}{g!} + 2\varphi(X) - 8g \log 2\pi.$$

Note, that $\varphi(X) = 0$ if $g = 1$. Next, we state some applications of the theorem. The first application is an explicit lower bound for $\delta(X)$ depending only on $g$.

**Corollary 1.** *Any compact and connected Riemann surface $X$ of genus $g \geq 1$ satisfies $\delta(X) > -2g \log 2\pi^4$.*

In the proof we apply the inequalities $\int_{\mathrm{Pic}_{g-1}(X)} \log\|\theta\| \frac{\nu^g}{g!} < -\frac{g}{4}\log 2$ and $\varphi(X) \geq 0$, where the former follows from Jensen's formula.

As another application, we obtain a canonical extension of the invariants $\delta$ and $\varphi$ to indecomposable principally polarised complex abelian varieties. De Jong introduced in [dJo10] the function $\eta = {}^{\mathrm{t}}(\theta_j)(\theta_{jk})^c(\theta_j)$ on $\mathbb{C}^g$, where $(\theta_j)$ is the vector of the first partial derivations of a theta function $\theta$ associated to a principally polarised complex abelian variety and $(\theta_{jk})$ is the matrix of its second partial derivations. In [dJo08] he also defined a real valued version $\|\eta\|$ on $\Theta$, the zero divisor of $\theta$, see also Section 1.1. We will deduce the following theorem from Theorem 1 and from a formula for $\delta(X)$ by de Jong [dJo08, Theorem 4.4].

**Theorem 2.** *For any compact and connected Riemann surface $X$ of genus $g \geq 2$, the invariant $\delta(X)$ satisfies*

$$\delta(X) = 2(g-7)\int_{\mathrm{Pic}_{g-1}(X)} \log\|\theta\| \frac{\nu^g}{g!} - 2\int_{\Theta} \log\|\eta\| \frac{\nu^{g-1}}{g!} - 4g\log 2\pi.$$

*Moreover, the invariant $\varphi(X)$ satisfies*

$$\varphi(X) = (g+5)\int_{\mathrm{Pic}_{g-1}(X)} \log\|\theta\| \frac{\nu^g}{g!} - \int_{\Theta} \log\|\eta\| \frac{\nu^{g-1}}{g!} + 2g\log 2\pi.$$

Here, $\Theta \subseteq \mathrm{Pic}_{g-1}(X)$ is the canonical divisor consisting of line bundles of degree $g-1$ having global sections. Also, we deduce an explicit formula for the Arakelov–Green function. For this purpose, we calculate the invariant $A(X)$ in Bost's formula for the Arakelov–Green function, see [Bos87].

**Theorem 3.** *For any compact and connected Riemann surface $X$ of genus $g \geq 2$ it holds*

$$\log G(P,Q) = \int_{\Theta+P-Q} \log\|\theta\| \frac{\nu^{g-1}}{g!} + \frac{1}{2g}\varphi(X) - \int_{\mathrm{Pic}_{g-1}(X)} \log\|\theta\| \frac{\nu^g}{g!}.$$

As a consequence of the theorem, we obtain the following upper bound for the Arakelov–Green function.

**Corollary 2.** *Let $X$ be any compact and connected Riemann surface of genus $g \geq 2$. The Arakelov–Green function is bounded by $\delta(X)$ in the following way:*

$$\sup_{P,Q\in X} \log G(P,Q) < \begin{cases} \frac{1}{4g}\delta(X) + 3g^3\log 2 & \text{if } g \leq 5, \\ \frac{2g+1}{48g}\delta(X) + 2g^3\log 2 & \text{if } g > 5. \end{cases}$$

Upper bounds for the Arakelov–Green function were already obtained by Merkl [EC11, Theorem 10.1] and Jorgenson–Kramer [JK06] by very different methods. But our bound seems to be more explicit and more natural in the sense of Arakelov theory. If $X$ is the modular curve $X_1(N)$, one can apply this bound to compute the complexity of an algorithm by Edixhoven for the computation of Galois representations associated to modular forms, see [EC11]. Indeed, $\delta(X_1(N))$ can be bounded polynomial in $N$, see for example [JK09, Remark 5.8] if $X_1(N)$ has genus $g \geq 2$, or [Jav14, Corollary 1.5.1] in general.

## Results on curves over number fields

To describe further applications of Theorem 1, let $C$ be again a smooth, projective and geometrically connected curve of genus $g \geq 2$ defined over a number field $K$ and $\mathscr{C}$ its minimal regular model over $B = \operatorname{Spec} \mathcal{O}_K$. We may assume that $\mathscr{C}$ is semi-stable. We set $d = [K : \mathbb{Q}]$. The stable Faltings height is defined by $h_F(C) = \frac{1}{d}\widehat{\deg} \det p_* \omega_{\mathscr{C}/B}$. For any geometric point $P$ of $C$ we denote by $h(P)$ the stable Arakelov height and by $h_{NT}(P)$ the Néron-Tate height; see Section 6.1 for the definitions of these heights. Let $W$ be the divisor of Weierstraß points of $C$. We will apply our lower bound of $\delta$ and an estimate of theta functions to a formula by de Jong [dJo09, Theorem 4.3] for the heights of the Weierstraß points of $C$. This yields the following bound.

**Proposition 1.** *The heights of the Weierstraß points of $C$ are bounded by*

$$\max_{P \in W} h(P) \leq \sum_{P \in W} h(P) < (6g^2 + 4g + 2) h_F(C) + 12 g^4 \cdot \log 2.$$

*In the summation over $W$ the Weierstraß points are counted with their multiplicity in $W$.*

De Jong obtained in [dJo04, Proposition 2.6.1], see also [EC11, Theorem 9.2.5], an estimate for the Arakelov intersection number of a torsion line bundle and an arbitrary geometric point on $C$. In the following situation, we can apply the bound in the above proposition and the bound of the Arakelov–Green function to make de Jong's estimate more explicit.

**Proposition 2.** *Let $W_1, \ldots, W_g$ be arbitrary and not necessary different Weierstraß points on $C$ and write $D$ for the effective divisor $\sum_{j=1}^{g} W_j$. Further, let $\mathcal{L}$ be any line bundle on $C$ of degree* $0$, *that is represented by a torsion point in* $\operatorname{Pic}^0(C)$ *and that satisfies* $\dim H^0(\mathcal{L}(D)) = 1$. *Write $D'$ for* *the unique effective divisor on $C$, such that $\mathcal{L} \cong \mathcal{O}_C(D' - D)$. Let $P \in C(\overline{K})$ be any geometric point of $C$. We may assume that $P, D, D'$ and $\mathcal{L}$ are defined over $K$. It holds*

$$\tfrac{1}{d}(D' - D, P) < 13g^4 \cdot h_F(C) + 28g^6 \cdot \log 2.$$

Our next application is motivated by Szpiro's small points conjecture [Szp85a]. This conjecture was proven by Javanpeykar and von Känel for cyclic covers of prime degree, see [JvK14]. Let $S$ be the set of finite places of $K$, where $C$ has bad reduction. We write $N_S = \prod_{v \in S} N_v$ and $D_K$ for the absolute value of the discriminant of $K$ over $\mathbb{Q}$. In the case $g = 2$, it is proven in [JvK14] that there are infinitely many geometric points $P$ of $C$ such that

$$\max(h_{NT}(P), h(P)) \leq \nu^{2d\nu}(N_S D_K)^\nu, \quad \nu = 10^5 d.$$

To prove this result, they first showed that if $C$ is a cyclic cover of prime degree, then it has infinitely many geometric points $P$ satisfying

$$\max(h_{NT}(P), h(P)) \leq \nu^{8^g d\nu}(N_S D_K)^\nu - \tfrac{1}{d} \sum_{\sigma\colon K \to \mathbb{C}} \delta(C_\sigma), \quad \nu = d(5g)^5. \quad (6)$$

Then they applied for $g = 2$ the lower bound $\delta(C_\sigma) \geq -186$. On combining our Corollary 1 with (6) we deduce the following generalization.

**Corollary 3.** *Suppose that $C$ is is a cyclic cover of prime degree. Then $C$ has infinitely many geometric points $P$, which satisfy*

$$\max(h_{NT}(P), h(P)) < \nu^{8^g d\nu}(N_S D_K)^\nu.$$

This improves the explicit bound in [JvK14], which depends exponentially on $N_S$ and $D_K$.

## Results on hyperelliptic curves over number fields

Next, we state lower and upper bounds of the Arakelov self-intersection number $(\omega_{\mathscr{C}/B}, \omega_{\mathscr{C}/B})$ for hyperelliptic curves. For general curves Faltings already proved in [Fal84, Theorem 5] that $(\omega_{\mathscr{C}/B}, \omega_{\mathscr{C}/B}) \geq 0$. Furthermore, Zhang [Zha92, Zha93] proved its strict positivity if $C$ has bad reduction at least at one finite place of $K$, and Moriwaki [Mor96, Mor97] gave an effective lower bound in this case. In general, Ullmo proved its strict positivity in [Ull98].

From now on, we assume $C$ to be hyperelliptic. Our proof allows us to deduce the following formula for $\delta$

$$\delta(C_\sigma) = -\tfrac{2(g-1)}{2g+1}\varphi(C_\sigma) - \tfrac{3g}{(2g+1)}\binom{2g}{g-1}^{-1} \log \|\Delta_g\|(C_\sigma) - 8g \log 2\pi. \quad (7)$$

This formula was already proved by de Jong [dJo13, Corollary 1.8] in a different way. Combining this formula with results by Kausz [Kau99] and Yamaki [Yam04], we can give an explicit description of $(\omega_{\mathscr{C}/B}, \omega_{\mathscr{C}/B})$ in terms of $\varphi(C_\sigma)$ and the geometry of the reduction of $C$ at the finite places of $K$. In particular, we deduce the following bounds for $(\omega_{\mathscr{C}/B}, \omega_{\mathscr{C}/B})$.

**Corollary 4.** *Let $C$ be any hyperelliptic curve as above. The Arakelov self-intersection number $(\omega_{\mathscr{C}/B}, \omega_{\mathscr{C}/B})$ is bounded in the following way:*

$$(\omega_{\mathscr{C}/B}, \omega_{\mathscr{C}/B}) \geq \tfrac{g-1}{2g+1}\left(\sum_{v\in|B|}\delta_v \log N_v + 2\sum_{\sigma\colon K\to\mathbb{C}}\varphi(C_\sigma)\right)$$

*and*

$$(\omega_{\mathscr{C}/B}, \omega_{\mathscr{C}/B}) \leq \tfrac{g-1}{2g+1}\left((3g+1)\sum_{v\in|B|}\delta_v \log N_v + 2\sum_{\sigma\colon K\to\mathbb{C}}\varphi(C_\sigma)\right).$$

Yamaki [Yam08, Corollary 4.2] proved an effective Bogomolov conjecture for hyperelliptic curves over function fields using an explicit expression for the self-intersection number of the canonical bundle. For hyperelliptic curves over number fields one can adopt this proof starting with our explicit expression for $(\omega_{\mathscr{C}/B}, \omega_{\mathscr{C}/B})$. Precisely, we obtain the following corollary.

**Corollary 5.** *Let $C$ be any hyperelliptic curve as above and $z$ any geometric point of $\mathrm{Pic}^0(C)$. There are only finitely many geometric points $P \in C(\overline{K})$ satisfying*

$$h_{NT}(((2g-2)P - K_C) - z) \leq \tfrac{(g-1)^2}{2g+1}\left(\tfrac{2g-5}{12gd}\sum_{v\in|B|}\delta_v \log N_v + \tfrac{1}{d}\sum_{\sigma\colon K\to\mathbb{C}}\varphi(C_\sigma)\right),$$

*where $K_C$ denotes the canonical bundle on $C$.*

We can deduce from Corollary 4 the upper bound

$$(\omega_{\mathscr{C}/B}, \omega_{\mathscr{C}/B}) < \tfrac{g-1}{2g+1}\left((3g+1)\sum_{v\in|B|}\delta_v \log N_v + \sum_{\sigma}\delta(C_\sigma) + 2gd\log 2\pi^4\right),$$

which is of the form (5) suggested by Parshin. This improves similar, but less explicit bounds by Kausz [Kau99, Corollary 7.8] and Maugeais [Mau03, Corollaire 2.11]. Nevertheless, this does not suffice to deduce any arithmetic consequences by the same methods as in [Par90] since we assume $C$ to be hyperelliptic.

The method of proof of Theorem 1 allows us moreover to establish the following generalization of Rosenhain's formula on $\theta$-derivatives. We denote by $\|J\|$ the derivative version of $\|\theta\|$ introduced by Guàrdia [Guà99, Definition 2.1].

**Theorem 4.** *Let $X$ be any hyperelliptic Riemann surface of genus $g \geq 2$ and denote by $W_1, \ldots, W_{2g+2}$ the Weierstraß points of $X$. For any permutation $\tau \in \mathrm{Sym}(2g+2)$ it holds*

$$\|J\|(W_{\tau(1)}, \ldots, W_{\tau(g)}) = \pi^g \prod_{j=g+1}^{2g+2} \|\theta\|(W_{\tau(1)} + \cdots + W_{\tau(g)} - W_{\tau(j)}).$$

This gives an absolute value answer to a conjecture by Guàrdia [Guà02, Conjecture 14.1].

# Main ideas of the proof

We describe the principal ideas of the proof of Theorem 1. The case $g = 1$ follows from Faltings' computations for elliptic curves in [Fal84, Section 7]. Hence, we can assume $g \geq 2$.

## Reduction to hyperelliptic Riemann surfaces

Consider

$$f(X) = \delta(X) + 24 \int_{\mathrm{Pic}_{g-1}(X)} \log \|\theta\| \frac{\nu^g}{g!} - 2\varphi(X) \tag{8}$$

as a real-valued function on $\mathcal{M}_g$, the moduli space of compact and connected Riemann surfaces of genus $g$. Theorem 1 asserts that we constantly have $f(X) \equiv -8g \log 2\pi$. We will reduce to prove Theorem 1 for hyperelliptic Riemann surfaces by showing that $f(X)$ is harmonic. Since there are no non-constant harmonic functions on $\mathcal{M}_g$, it follows that $f(X)$ is constant. Since for any $g \geq 2$ there exists at least one hyperelliptic Riemann surface of genus $g$, it is then enough to compute $f(X)$ if $X$ is hyperelliptic.

To prove that $f(X)$ is harmonic on $\mathcal{M}_g$, we apply the Laplace operator $\partial\overline{\partial}$ on $\mathcal{M}_g$ to the terms in (8). For $\varphi(X)$ and $\delta(X)$ we have an expression for the resulting forms in terms of the canonical forms $e_1^A$, $\int_{\pi_2} h^3$ and $\omega_{\mathrm{Hdg}}$ (see Section 4.1) on $\mathcal{M}_g$ by de Jong [dJo14b]. To calculate the application of $\partial\overline{\partial}$ to the integral in (8), we apply the Laplace operator on the universal abelian variety with level 2 structure to $\log\|\theta\|$ and we pull back the integral to the $(g+1)$-th power of the universal Riemann surface with level 2 structure $\mathcal{X}_g \to \mathcal{M}_g[2]$. The pullback can be expressed in terms of Deligne pairings by a result due to de Jong [dJo14b, Proposition 6.3]. The main difficulty is to express the first Chern form of the $(g+1)$-th power in the sense of Deligne pairings of the line bundle

$$\bigotimes_{j=1}^{g} pr_j^* T_{\mathcal{X}_g/\mathcal{M}_g[2]} \otimes \bigotimes_{j=1}^{g} pr_{j,g+1}^* \mathcal{O}(\Delta)^\vee \otimes \bigotimes_{j<k}^{g} pr_{j,k}^* \mathcal{O}(\Delta)$$

on $\mathcal{X}_g^{g+1}$ in terms of the forms $e_1^A$, $\int_{\pi_2} h^3$ and $e^A$. Here, $T_{\mathcal{X}_g/\mathcal{M}_g[2]}$ denotes the relative tangent bundle, $\Delta \subseteq \mathcal{X}_g^2$ is the diagonal and $pr_j$ and $pr_{j,k}$ denote the projections to the respective factors of $\mathcal{X}_g^{g+1}$. We solve this problem by associating graphs to the terms in the expansion of the power, which we can classify and count.

## The hyperelliptic case

To prove Theorem 1 for any hyperelliptic Riemann surface $X$ of genus $g \geq 2$, we generalize Bost's formula (3) to hyperelliptic Riemann surfaces of genus $g \geq 2$. The generalized formula states

$$\delta(X) = -\tfrac{8(g-1)}{g} \int_{\mathrm{Pic}_{g-1}(X)} \log\|\theta\| \tfrac{\nu^g}{g!} - \tbinom{2g}{g-1}^{-1} \log\|\Delta_g\|(X) - 8g\log 2\pi. \quad (9)$$

We will see later in Section 4.4 that we can canonically define the invariant $\|\Delta_g\|(X)$ also for non-hyperelliptic Riemann surfaces, but formula (9) will not be true for general Riemann surfaces. The main ingredient of the proof of formula (9) is the following formula

$$\int_{\mathrm{Pic}_{g-1}(X)} \log\|\theta\|^{g-1} \tfrac{\nu^g}{g!} = \int_{X^g} \log \tfrac{\|\theta\|(P_1+\dots+P_g-Q)^g}{\|\theta\|(gP_1-Q)} \mu(P_1)\dots\mu(P_g). \quad (10)$$

The proof of (10) consists essentially of two steps. The first step is to decompose the function $\log\|\theta\|$ into a sum of Arakelov–Green functions and an additional invariant. This step generalizes the decomposition in [BMM90, A.1.] for $g=2$ to arbitrary hyperelliptic Riemann surfaces. In the second step, we use the decomposition of the first step to compare the pullbacks of the integral of $\log\|\theta\|$ on $\mathrm{Pic}_{g-1}(X)$ under the maps

$$\Phi\colon X^g \to \mathrm{Pic}_{g-1}(X), \quad (P_1,\dots,P_g) \mapsto (P_1+\dots+P_g-Q),$$
$$\Psi\colon X^g \to \mathrm{Pic}_{g-1}(X), \quad (P_1,\dots,P_g) \mapsto (2P_1+P_2+\dots+P_{g-1}-P_g).$$

In this step we also obtain a connection of the integrals of $\log\|\theta\|$ to the invariant $\varphi(X)$.

To connect formula (10) with Faltings' $\delta$-invariant, we compare two invariants obtained by the function $\|J\|$: The iterated integral over $X^g$ of its logarithm and the product of its values in Weierstraß points. Then we apply Guàrdia's expression for $\delta(X)$ in [Guà99, Proposition 1.1] to connect the first invariant with (10) and $\delta(X)$. De Jong proved in [dJo07, Theorem 9.1] that the second invariant is essentially $\|\Delta_g\|(X)$. This together with de Jong's formula (4) leads to formula (9) and also to the formula in Theorem 1 by the connection of the integrals of $\log\|\theta\|$ to $\varphi(X)$.

We remark, that there is an alternative way to compute the constant $f(X)$: Consider a family of Riemann surfaces $\mathscr{X}_t$ of genus $g \geq 2$ degenerating to a singular complex curve $\mathscr{X}_0$ consisting of two Riemann surfaces $X_1$ and $X_2$ of genus $g-1$ respectively 1 meeting in one point. For this family the integral in (8) degenerates nicely and the asymptotic behaviour of $\delta(\mathscr{X}_t)$ and $\varphi(\mathscr{X}_t)$ were studied by Wentworth [Wen91] and de Jong [dJo14a]. Using their results, one can deduce that $f(\mathscr{X}_t)$ degenerates to $f(X_1)+f(X_2)$ in this family. Hence, one obtains $f(X) = -8g\log 2\pi$ by induction. However, the methods of our proof for hyperelliptic Riemann surfaces are of independent interest. For example, they also prove formulas (7) and (9) and Theorem 4.

# Overview

In the following, we explain the structure of this thesis. In the first Chapter we define all required invariants of abelian varieties and Riemann surfaces. In Chapter 2 we study certain integrals of $\log\|\theta\|$ and of the Arakelov–Green function. The third chapter deals with the proof of Theorem 1 for hyperelliptic Riemann surfaces. First, we introduce our decomposition of $\log\|\theta\|$ in Section 3.1, and we proof equation (10) in Section 3.2. In the subsequent section we obtain Theorem 1 for hyperelliptic Riemann surfaces, and we give some consequences and examples. In Section 3.4 we apply our decomposition of $\log\|\theta\|$ to prove Theorem 4.

In Chapter 4 we prove Theorem 1 in general. For this purpose, we discuss the forms obtained by applying the Laplace operator $\partial\overline{\partial}$ on $\mathcal{M}_g$ to $\varphi(X)$, $\delta(X)$ and $H(X)$ in Section 4.1. To compare the latter one with the former ones, we introduce the Deligne pairing in Section 4.2, and we calculate the expansion of the power in the sense of Deligne pairings of a certain line bundle using graphs in Section 4.3. In Section 4.4 we conclude our main result Theorem 1, and we deduce Corollary 1. After we bound the function $\|\theta\|$ in Section 4.5, we study the Arakelov–Green function in Section 4.6, where we obtain Theorem 3 and Corollary 2.

In Chapter 5 we consider indecomposable principally polarised complex abelian varieties. We prove Theorem 2 in Section 5.1. This yields a canonical extension of $\delta$ and $\varphi$ to indecomposable principally polarised complex abelian varieties. We discuss some of their asymptotic behaviours in Section 5.2. In the last chapter we apply our results to the arithmetic situation of Arakelov theory. In particular, we prove Propositions 1 and 2 and Corollary 3 in Section 6.1 and Corollaries 4 and 5 in Section 6.2.

## Acknowledgement

First of all, I would like to thank my advisor Professor Gerd Faltings for introducing me into Arakelov theory and for his suggestion to study the $\delta$-invariant. I also thank him for his highly enriching lectures at the University of Bonn. Further, I would like to thank the committee members Professor Michael Rapoport, Professor Sergio Conti and Professor Norbert Blum. My gratitude also goes to Rafael von Känel and Robin de Jong for useful discussions. I thank Christian Kaiser for spending many hours to discuss various topics in arithmetic geometry. I thank the Max Planck Society and the IMPRS for financial support and I also thank the stuff of the MPIM Bonn for their hospitality.

# Chapter 1

# Invariants

We give the definitions of the invariants appearing in the following chapters and we state some of their properties and relations.

## 1.1 Invariants of abelian varieties

In this section we define some invariants of abelian varieties. Let $(A,\Theta)$ be any principally polarised complex abelian variety of dimension $g \geq 1$, where $\Theta \subseteq A$ denotes a divisor, such that $\mathcal{O}(\Theta)$ is an ample and symmetric line bundle. The principal polarisation of $(A,\Theta)$ determines the divisor class of $\Theta$ only up to a translation by a 2-torsion point. There exists a complex and symmetric $g \times g$ matrix $\Omega$ with positive definite imaginary part $Y = \mathrm{Im}\ \Omega$, such that $A = \mathbb{C}^g/(\mathbb{Z}^g + \Omega\mathbb{Z}^g)$ and $\Theta$ is the zero divisor of the function

$$\theta\colon \mathbb{C}^g \to \mathbb{C}, \quad z \mapsto \theta(z) = \theta(\Omega; z) = \sum_{n\in\mathbb{Z}^g} \exp(\pi i {}^t n \Omega n + 2\pi i {}^t n z),$$

see for example [BL04, Chapter 8].

Since we have $\theta(z+m+n\Omega) = \exp(-\pi i {}^t n\Omega n - 2\pi i {}^t n z)\theta(z)$ for $m, n \in \mathbb{Z}^g$, we obtain a well-defined, real-valued function $\|\theta\|\colon A \to \mathbb{R}_{\geq 0}$ by

$$\|\theta\|(z) = \|\theta\|(\Omega; z) = \det(Y)^{1/4} \exp(-\pi {}^t(\mathrm{Im}\ z) Y^{-1} (\mathrm{Im}\ z)) \cdot |\theta|(z).$$

We associate to $(A,\Theta)$ the canonical $(1,1)$-form

$$\nu = \nu_{(A,\Theta)} = \tfrac{i}{2} \sum_{j,k=1}^{g} (Y^{-1})_{jk} dZ_j \wedge d\overline{Z}_k,$$

where $Z_1, \ldots, Z_g$ are coordinates in $\mathbb{C}^g$. This form is translation-invariant. The function $\|\theta\|$ could also be defined as the unique function $\|\theta\|\colon A \to \mathbb{R}_{\geq 0}$ satisfying:

($\theta$1) The zero divisor of $\|\theta\|$ is $\Theta$.

($\theta$2) For $z \notin \Theta$ its curvature is given by $\partial\bar{\partial} \log \|\theta\|(z)^2 = 2\pi i \nu$.

($\theta$3) The function is normed by $\frac{1}{g!}\int_A \|\theta\|^2 \nu^g = 2^{-g/2}$.

For a calculation of the third property see [BL04, Proposition 8.5.6]. In particular, the function $\|\theta\|$ depends on the choice of $\Theta$ for a principally polarised complex abelian variety. Further, we define the following invariant

$$H(A,\Theta) = \tfrac{1}{g!}\int_A \log \|\theta\| \nu^g.$$

If $(A,\Theta)$ is the Jacobian variety of a Riemann surface $X$ of genus $g = 2$, this definition coincides with the definition of $\log \|H\|(X)$ in [Bos87]. The invariant $H(A,\Theta)$ is bounded from above.

**Proposition 1.1.1.** *Any principally polarised complex abelian variety $(A,\Theta)$ of dimension $g \geq 1$ satisfies $H(A,\Theta) < -\frac{g}{4}\log 2$.*

*Proof.* Since $\int_A \nu^g = g!$, Jensen's inequality and ($\theta$3) give

$$2H(A,\Theta) = \tfrac{1}{g!}\int_A \log \|\theta\|^2 \nu^g < \log\left(\tfrac{1}{g!}\int_A \|\theta\|^2\nu^g\right) = -\tfrac{g}{2}\log 2.$$

□

We obtain another function $\eta\colon \mathbb{C}^g \to \mathbb{C}$ by considering certain partial derivations of $\theta$:

$$\eta(z) = \det \begin{pmatrix} \frac{\partial^2\theta}{\partial Z_1 \partial Z_1}(z) & \cdots & \frac{\partial^2\theta}{\partial Z_1 \partial Z_g}(z) & \frac{\partial\theta}{\partial Z_1}(z) \\ \vdots & \ddots & \vdots & \vdots \\ \frac{\partial^2\theta}{\partial Z_g \partial Z_1}(z) & \cdots & \frac{\partial^2\theta}{\partial Z_g \partial Z_g}(z) & \frac{\partial\theta}{\partial Z_g}(z) \\ \frac{\partial\theta}{\partial Z_1}(z) & \cdots & \frac{\partial\theta}{\partial Z_g}(z) & 0 \end{pmatrix},$$

see also [dJo10]. Further, we get a real-valued variant $\|\eta\|\colon \Theta \to \mathbb{R}_{\geq 0}$ by

$$\|\eta\|(z) = \det(Y)^{(g+5)/4}\exp(-\pi(g+1)^{\mathrm{t}}(\mathrm{Im}\ z)Y^{-1}(\mathrm{Im}\ z)) \cdot |\eta|(z),$$

see also [dJo08]. The function $\eta$ is identically zero on $\Theta$ if and only if $(A,\Theta)$ is decomposable, see [dJo10, Corollary 3.2]. We set

$$\Lambda(A,\Theta) = \tfrac{1}{g!}\int_\Theta \log \|\eta\| \nu^{g-1}$$

if $(A,\Theta)$ is indecomposable.

By definition, the invariants $H(A,\Theta)$ and $\Lambda(A,\Theta)$ are invariant under translation by 2-torsion points of the divisor $\Theta$. Hence, we can indeed consider them as invariants of (indecomposable) principally polarised complex abelian varieties.

## 1.2 Invariants of Riemann surfaces

In this section we define some invariants of Riemann surfaces. Let $X$ be a compact and connected Riemann surface of genus $g \geq 1$. We choose a symplectic basis of homology $A_i, B_i \in H_1(X, \mathbb{Z})$, that means we have the intersection pairings $(A_i.A_j) = (B_i.B_j) = 0$ and $(A_i.B_j) = \delta_{ij}$ for all $i, j$, where $\delta_{ij}$ denotes the Kronecker symbol. Further, we choose a basis of one forms $\omega_1, \ldots, \omega_g \in H^0(X, \Omega^1_X)$, such that $\int_{A_j} \omega_i = \delta_{ij}$. We associate to $X$ its period matrix $\Omega = \Omega_X$, which is given by $\Omega_{ij} = \int_{B_i} \omega_j$. It is symmetric and has positive definite imaginary part denoted by $Y = \operatorname{Im} \Omega$. The Jacobian of $X$, denoted by $\operatorname{Jac}(X)$, is the principally polarised abelian variety associated to $\Omega$. In the following, we shortly write $\nu = \nu_{\operatorname{Jac}(X)}$. For a fixed base point $Q \in X$ the Abel–Jacobi map is given by the embedding $I\colon X \to \operatorname{Jac}(X)$, $P \mapsto (\int_Q^P \omega_1, \ldots, \int_Q^P \omega_g)$. We define the canonical $(1,1)$ form $\mu$ on $X$ by $\mu = \frac{1}{g} I^* \nu$, which has volume $\int_X \mu = 1$. Further, we denote the canonical bundle on $X$ by $K_X$.

There is a unique theta characteristic $\alpha_X$, that is $2\alpha_X = K_X$, which gives an isomorphism

$$\operatorname{Pic}_{g-1}(X) \xrightarrow{\sim} \operatorname{Jac}(X), \quad \mathcal{L} \mapsto (\mathcal{L} - \alpha_X), \tag{1.2.1}$$

such that $\|\theta\|(\Omega; \mathcal{L} - \alpha_X) = 0$ if and only if $H^0(X, \mathcal{L}) \neq 0$, see for example [Mum83, Corollary II.3.6]. We simply write $\|\theta\|(D) = \|\theta\|(\mathcal{O}(D) - \alpha_X)$ for a divisor $D$ of degree $g-1$. It follows, that $\|\theta\|(D) = 0$ if and only if $D$ is linearly equivalent to an effective divisor. We denote $\Theta \subseteq \operatorname{Pic}_{g-1}(X)$ for the divisor in $\operatorname{Pic}_{g-1}(X)$ defined by the zeros of $\|\theta\|$. Equivalently, $\Theta$ is given by the line bundles of degree $g-1$ having global sections. For any effective divisor $D$ of degree $g-1$ we also write $\|\eta\|(D) = \|\eta\|(\mathcal{O}(D) - \alpha_X)$. Since the divisor $\Theta \subseteq \operatorname{Pic}_{g-1}(X)$ is canonical, the functions $\|\theta\|$ and $\|\eta\|$ on $\operatorname{Pic}_{g-1}(X)$ do not depend on the choice of the period matrix $\Omega$.

We set $H(X) = H(\operatorname{Jac}(X))$ and $\Lambda(X) = \Lambda(\operatorname{Jac}(X))$. Another invariant $S(X)$ of $X$ was defined by de Jong in [dJo05a, Section 2]. It satisfies $\log S(X) = -\int_X \log \|\theta\|(gP - Q)\mu(P)$. We generalize this to a family of invariants given by

$$S_k(X) = \int_{X^k} \log \|\theta\|((g-k+1)P_1 + P_2 + \cdots + P_k - Q)\mu(P_1)\ldots\mu(P_k)$$

for every $1 \leq k \leq g$, where $k$ stands for the dimension of the integral. In particular, we have $S_1(X) = -\log S(X)$. We will prove a relation between $H(X)$, $S_1(X)$ and $S_g(X)$ for hyperelliptic Riemann surfaces in Section 3.2.

Another way to build an invariant from the function $\theta$ is to consider the integral of certain derivatives. For this purpose, we define as in [Guà99, Definition 2.1]

$$\|J\|(P_1,\dots,P_g) = \det(Y)^{(g+2)/4} \exp\left(-\pi \sum_{k=1}^{g} {}^{\mathrm{t}}y_k Y^{-1} y_k\right) \left|\det\left(\tfrac{\partial\theta}{\partial Z_l}(w_k)\right)\right|,$$

where $P_1,\dots,P_g$ denote arbitrary points of $X$, $w_k \in \mathbb{C}^g$ is a lift of the divisor class $(P_1+\dots+P_g-P_k-\alpha_X) \in \mathrm{Jac}(X)$ and $y_k$ is the imaginary part of $w_k$. To get an invariant we set

$$B(X) = \int_{X^g} \log\|J\|(P_1,\dots,P_g)\mu(P_1)\dots\mu(P_g).$$

For hyperelliptic Riemann surfaces we will give a relation between $B(X)$ and $S_g(X)$ in Section 3.1.

We define the Arakelov–Green function $G\colon X^2 \to \mathbb{R}_{\geq 0}$ as the unique function satisfying the following conditions:

(G1) We have $G(P,Q) > 0$ for $P \neq Q$. For a fixed $Q \in X$, $G(P,Q)$ has a simple zero in $P = Q$.

(G2) For $P \neq Q$ the curvature with respect to the first coordinate is given by $\partial_P\bar{\partial}_P \log G(P,Q)^2 = 2\pi i\mu(P)$.

(G3) It is normalized by $\int_X \log G(P,Q)\mu(P) = 0$.

One can check that $G(P,Q) = G(Q,P)$. For shorter notation we write $g(P,Q) = \log G(P,Q)$. Bost has shown in [Bos87, Proposition 1], that there is an invariant $A(X)$, such that

$$g(P,Q) = \tfrac{1}{g!}\int_{\Theta+P-Q} \log\|\theta\|\nu^{g-1} + A(X). \tag{1.2.2}$$

We will give an explicit expression for the invariant $A(X)$ in Section 4.6. Another natural invariant defined by the Arakelov–Green function is its supremum $\sup_{P,Q\in X} g(P,Q)$. We will bound it by more explicit invariants also in Section 4.6.

Next, we recall the definition of Faltings' $\delta$ invariant in [Fal84, p. 402]. Let $\psi_1,\dots,\psi_g$ be another basis of $H^0(X,\Omega_X^1)$, which is orthonormal with respect to the inner product

$$\langle\psi,\psi'\rangle = \tfrac{i}{2}\int_X \psi\wedge\psi'. \tag{1.2.3}$$

Then the defining equation for $\delta(X)$ is

$$\|\theta\|(P_1+\cdots+P_g-Q)=\exp\left(-\tfrac{1}{8}\delta(X)\right)\cdot\frac{\|\det(\psi_j(P_k))\|_{\mathrm{Ar}}}{\prod_{j<k}G(P_j,P_k)}\cdot\prod_{j=1}^{g}G(P_j,Q),$$

where $P_1,\dots,P_g,Q$ are pairwise different points, such that the class of the divisor $(P_1+\cdots+P_g-Q)$ lies not in $\Theta$ and the Arakelov norm $\|\cdot\|_{\mathrm{Ar}}$ of holomorphic one forms is induced by

$$\|dz\|_{\mathrm{Ar}}(P)=\lim_{Q\to P}\frac{|z(Q)-z(P)|}{G(P,Q)},$$

where $z\colon U\to\mathbb{C}$ is a local coordinate of a neighbourhood $P\in U\subseteq X$. This invariant plays an important role in Arakelov theory. For example, it is up to a constant the $\delta$ in the arithmetic Noether formula for the archimedean places. In Section 4.4 we will obtain a new expression and a lower bound only in terms of $g$ for $\delta(X)$.

In [Guà99, Proposition 1.1] Guàrdia gave the following expression

$$\|\theta\|(P_1+\cdots+P_g-Q)^{g-1}=\exp\left(\tfrac{1}{8}\delta(X)\right)\frac{\|J\|(P_1,\dots,P_g)}{\prod_{j<k}G(P_j,P_k)}\prod_{j=1}^{g}G(P_j,Q)^{g-1}. \tag{1.2.4}$$

Taking logarithm and integrating with $\mu(P_1)\dots\mu(P_g)$ gives

$$\delta(X)=8(g-1)S_g(X)-8B(X). \tag{1.2.5}$$

We state another formula for $\delta(X)$ by de Jong. For this purpose, let $\Theta^{sm}$ be the smooth part of $\Theta\subseteq\mathrm{Pic}_{g-1}(X)$. Every divisor $D\in\Theta^{sm}$ has a unique representation $D=P_1+\cdots+P_{g-1}$ for some points $P_1,\dots,P_{g-1}$ on $X$. By Riemann–Roch, the involution

$$\sigma\colon\mathrm{Pic}_{g-1}(X)\to\mathrm{Pic}_{g-1}(X),\quad D\to K_X-D$$

induces an involution on $\Theta^{sm}$. For effective divisors $D=P_1+\cdots+P_r$ and $E=Q_1+\cdots+Q_s$ we define the Arakelov–Green function by

$$G(D,E)=\prod_{j=1}^{r}\prod_{k=1}^{s}G(P_j,Q_k).$$

For any $D\in\Theta^{sm}$ and any different points $Q,R\in X$, such that $Q$ (respectively R) is not contained in the unique expression of $D$ (respectively $\sigma(D)$) as sum of $g-1$ points, we have by [dJo08, Theorem 4.4]

$$\|\eta\|(D)=\exp\left(-\tfrac{1}{4}\delta(X)\right)G(D,\sigma(D))\left(\frac{\|\theta\|(D+R-Q)}{G(R,Q)G(D,Q)G(\sigma(D),R)}\right)^{g-1}.$$

Write $D = P_1 + \cdots + P_{g-1}$. If we take the product of the $g-1$ equations obtained by putting $R = P_j$ for each $j \leq g-1$ in the above equation, we get

$$\|\eta\|(P_1 + \cdots + P_{g-1}) = \exp\left(-\tfrac{1}{4}\delta(X)\right) \prod_{j=1}^{g-1} \frac{\|\theta\|(P_1 + \cdots + P_{g-1} + P_j - Q)}{G(P_j, Q)^g}. \tag{1.2.6}$$

Next, we define the Kawazumi–Zhang invariant $\varphi(X)$, which was introduced and studied independently in [Kaw08] and [Zha10]. For this purpose, we consider the diagonal divisor $\Delta \subseteq X^2$. We have a canonical hermitian metric on $\mathcal{O}_{X^2}(\Delta)$ by $\|1\|(P_1, P_2) = G(P_1, P_2)$, where 1 is the canonical section of $\mathcal{O}_{X^2}(\Delta)$. We denote by $h_\Delta$ the curvature form of $\mathcal{O}_{X^2}(\Delta)$. It can be given explicitly by

$$h_\Delta(P_1, P_2) = \mu(P_1) + \mu(P_2) - \tfrac{i}{2} \sum_{k=1}^{g} \left(\psi_k(P_1) \wedge \bar{\psi}_k(P_2) + \psi_k(P_2) \wedge \bar{\psi}_k(P_1)\right), \tag{1.2.7}$$

see [Ara74, Proposition 3.1]). We define $\varphi(X)$ by

$$\varphi(X) = \int_{X^2} g(P_1, P_2) h_\Delta^2(P_1, P_2), \tag{1.2.8}$$

see [Zha10, Proposition 2.5.3]. It is not difficult to prove $\varphi(X) = 0$ for $g = 1$ and that we have the lower bound

$$\varphi(X) > 0 \tag{1.2.9}$$

for $g \geq 2$, see [Kaw08, Corollary 1.2] or [dJo14b, Proposition 4.2].

## 1.3 Invariants of hyperelliptic Riemann surfaces

We consider the more special case of hyperelliptic Riemann surfaces. Let $X$ be any hyperelliptic Riemann surface of genus $g \geq 2$. That means, there are pairwise different complex numbers $a_1, \ldots, a_{2g+1} \in \mathbb{C}$, such that $X$ is given by the equation

$$y^2 = (x - a_1) \cdot (x - a_2) \cdot \ldots \cdot (x - a_{2g+1}) (=: f(x)) \tag{1.3.1}$$

and the unique point at infinity, denoted by $\infty \in X$. There is a canonical involution induced by $y \mapsto -y$, which we denote by $\sigma\colon X \to X$. The fixed points of $\sigma$ are the Weierstraß points of $X$. They bijectively correspond to the points $x = a_1, \ldots, x = a_{2g+1}$ and the point $\infty$. We denote them by $W_1, \ldots, W_{2g+2}$, where $W_{2g+2} = \infty$. For the symplectic basis of homology $A_1, \ldots, A_g, B_1, \ldots, B_g$ we choose the canonical one, see [Mum84, Chapter IIIa, §5].

For hyperelliptic Riemann surfaces we define the Petersson norm of the modular discriminant $\|\varphi_g\|(X)$. For every $\eta = \begin{bmatrix} \eta' \\ \eta'' \end{bmatrix}$ with $\eta', \eta'' \in \frac{1}{2}\mathbb{Z}^g$ we set

$$\theta[\eta](z) = \exp\left(\pi i {}^{\mathrm{t}}\eta' \Omega \eta' + 2\pi i {}^{\mathrm{t}}\eta'(\eta'' + z)\right) \theta\left(\Omega\eta' + \eta'' + z\right).$$

Further, as in [Mum84, Chapter IIIa, Definition 5.7] we define

$$\eta_{2k-1} = \begin{bmatrix} {}^{\mathrm{t}}(0, \ldots, 0, \frac{1}{2}, 0, \ldots, 0) \\ {}^{\mathrm{t}}(\frac{1}{2}, \ldots, \frac{1}{2}, 0, 0, \ldots, 0) \end{bmatrix} \text{ for } 1 \leq k \leq g+1,$$

$$\eta_{2k} = \begin{bmatrix} {}^{\mathrm{t}}(0, \ldots, 0, \frac{1}{2}, 0, \ldots, 0) \\ {}^{\mathrm{t}}(\frac{1}{2}, \ldots, \frac{1}{2}, \frac{1}{2}, 0, \ldots, 0) \end{bmatrix} \text{ for } 1 \leq k \leq g,$$

where the non-zero entry in the top row occurs in the $k$-th position. For a subset $S \subseteq \{1, \ldots, 2g+1\}$ we set $\eta_S = \sum_{k \in S} \eta_k (\mathrm{mod}\ 1)$. We denote by $\mathcal{T}$ the collection of all subsets of $\{1, \ldots, 2g+1\}$ of cardinality $g+1$. Further, we set $U = \{1, 3, \ldots, 2g+1\}$ and we write $\circ$ for the symmetric difference.

We define the Petersson norm of the modular discriminant of $X$ by

$$\|\varphi_g\|(X) = (\det Y)^{2\binom{2g+1}{g+1}} \prod_{T \in \mathcal{T}} |\theta[\eta_{T \circ U}](0)|^8 .$$

Further, we denote a modified version by

$$\|\Delta_g\|(X) = 2^{-4(g+1)\binom{2g}{g-1}} \|\varphi_g\|(X).$$

For a discussion on the relation between $\|\varphi_g\|(X)$ and the discriminant of the polynomial $f$ in (1.3.1) we refer to Lockhart [Loc94].

As a direct consequence of the correspondence in [Mum84, IIIa Proposition 6.2] we obtain the following identity

$$\|\varphi_g\|(X) = \prod_{\{j_1, \ldots, j_{g+1}\} \in \mathcal{T}} \|\theta\|(W_{j_1} + \cdots + W_{j_g} - W_{j_{g+1}})^8. \tag{1.3.2}$$

Since we can choose every Weierstraß point to be the point at infinity and the invariant $\|\varphi_g\|(X)$ does not depend on this choice, we get by taking the product over all these choices

$$\|\varphi_g\|(X) = \prod_{\{j_1, \ldots, j_{g+1}\} \in \mathcal{U}_{g+1}} \|\theta\|(W_{j_1} + \cdots + W_{j_g} - W_{j_{g+1}})^4, \tag{1.3.3}$$

where we denote by $\mathcal{U}_k$ the collection of all subsets of $\{1, \ldots, 2g+2\}$ of cardinality $k$. Due to de Jong we have the relations

$$\delta(X) = \tfrac{4(g-1)}{g^2} S_1(X) - \tfrac{3g-1}{2g} \tbinom{2g}{g-1}^{-1} \log \|\Delta_g\|(X) - 8g \log 2\pi, \tag{1.3.4}$$

see [dJo05b, Corollary 1.7], and

$$\prod_{\{j_1,\ldots,j_g\} \in \mathcal{U}_g} \|J\|(W_{j_1}, \ldots, W_{j_g}) = \pi^{\binom{2g+2}{g} g} \cdot \|\varphi_g\|(X)^{(g+1)/4}, \tag{1.3.5}$$

see [dJo07, Theorem 9.1].

# Chapter 2

# Integrals

We give some relations between different integrals of the function $\log \|\theta\|$ and the Arakelov-Green function. Let $X$ be any compact and connected Riemann surface of genus $g \geq 2$ throughout this chapter.

## 2.1 Integrals of theta functions

In this section we establish two ways to write the invariant $H(X)$ as an integral over $X^g$. For a fixed base point $Q \in X$ we define the maps

$$\Phi\colon X^g \to \mathrm{Pic}_{g-1}(X), \quad (P_1, \dots, P_g) \mapsto (P_1 + \cdots + P_g - Q),$$
$$\Psi\colon X^g \to \mathrm{Pic}_{g-1}(X), \quad (P_1, \dots, P_g) \mapsto (2P_1 + P_2 + \cdots + P_{g-1} - P_g).$$

**Proposition 2.1.1.** *The maps $\Phi$ and $\Psi$ are smooth and surjective. Moreover, $\Phi$ is generically of degree $g!$, $\Psi$ is generically of degree $4g!$, and the pullbacks of the volume form $\nu^g$ satisfy $\Psi^*\nu^g = 4\Phi^*\nu^g$.*

*Proof.* The maps are defined as linear combinations of the Abel–Jacobi map. Hence, they are smooth. Jacobi's inversion theorem gives the surjectivity of $\Phi$, see for example [FK80, III.6]. If we have divisors $P_1 + \cdots + P_g - Q$ and $R_1 + \cdots + R_g - Q$ representing the same class in $\mathrm{Pic}_{g-1}(X) \setminus \Theta$, then

$$\|\theta\|(P_1 + \cdots + P_g - Q) = \|\theta\|(R_1 + \cdots + R_g - Q)$$

has zeros in $P_1, \dots, P_g, R_1, \dots, R_g$ as a function in $Q$. But it has exactly $g$ zeros counted with multiplicities, see for example [Mum83, Theorem II.3.1]. Hence, the $g$-tuples $(P_1, \dots, P_g)$ and $(R_1, \dots, R_g)$ coincide up to order. Thus, $\Phi$ is generically of degree $g!$.

For the pullbacks of $dZ_k$ we get

$$\Phi^*(dZ_k) = \sum_{j=1}^{g} \omega_k(P_j) \quad \text{and} \quad \Psi^*(dZ_k) = 2\omega_k(P_1) - \omega_k(P_g) + \sum_{j=2}^{g-1} \omega_k(P_j).$$

Therefore, $\Phi^*\nu^g$ is a linear combination of terms of the form

$$\omega_{\rho(1)}(P_1) \wedge \bar{\omega}_{\tau(1)}(P_1) \wedge \cdots \wedge \omega_{\rho(g)}(P_g) \wedge \bar{\omega}_{\tau(g)}(P_g),$$

for two permutations $\rho, \tau \in \mathrm{Sym(g)}$. But $\Psi^*\nu^g$ is the same linear combination in the terms

$$2\omega_{\rho(1)}(P_1) \wedge 2\bar{\omega}_{\tau(1)}(P_1) \wedge \omega_{\rho(2)}(P_2) \wedge \cdots \wedge \bar{\omega}_{\tau(g-1)}(P_{g-1}) \wedge -\omega_{\rho(g)}(P_g) \wedge -\bar{\omega}_{\tau(g)}(P_g).$$

Thus, we have $\Psi^*\nu^g = 4\Phi^*\nu^g$.

Since $\Psi^*\nu^g$ is non-zero, the image of $\Psi$ has to be of dimension $g$. Hence, we have $\Psi(X^g) = \mathrm{Pic}_{g-1}(X)$, since the image is compact and $\mathrm{Pic}_{g-1}(X)$ is an abelian variety. The degree of $\Psi$ is $4g!$ by $\Psi^*\nu^g = 4\Phi^*\nu^g$. □

We can now compute $H(X)$ by pulling back the integral by $\Phi$

$$H(X) = \tfrac{1}{(g!)^2} \int_{X^g} \log \|\theta\|(P_1 + \cdots + P_g - Q)\Phi^*\nu^g \tag{2.1.1}$$

and by pulling back the integral by $\Psi$

$$\begin{aligned} H(X) &= \tfrac{1}{4(g!)^2} \int_{X^g} \log \|\theta\|(2P_1 + P_2 + \cdots + P_{g-1} - P_g)\Psi^*\nu^g \qquad (2.1.2) \\ &= \tfrac{1}{(g!)^2} \int_{X^g} \log \|\theta\|(2P_1 + P_2 + \cdots + P_{g-1} - P_g)\Phi^*\nu^g, \end{aligned}$$

see for example [DFN85, Theorem 14.1.1].

## 2.2 Integrals of the Arakelov–Green function

We compare integrals of the Arakelov–Green function with respect to the form $\Phi^*\nu^g$ with the Kawazumi–Zhang invariant $\varphi$. First, we need a general lemma. For $k \leq g$ and points $R_1, \ldots, R_{g-k}, Q \in X$ we define the map

$$\begin{aligned} \Phi_k \colon X^k &\to \mathrm{Pic}_{g-1}(X), \\ (P_1, \ldots, P_k) &\mapsto P_1 + \cdots + P_k + R_1 + \cdots + R_{g-k} - Q. \end{aligned}$$

The pullbacks of $dZ_l$ are $\Phi_k^*(dZ_l) = \sum_{j=1}^{k} \omega_l(P_j)$. In particular, we have $g\mu = I^*\nu = \Phi_1^*\nu$. For the relation to $\Phi^*\nu^g$ we have the following lemma.

**Lemma 2.2.1.** *The integral of $\Phi^*\nu^g$ over the variables $P_{k+1},\dots,P_g$ gives the following multiple of the form $\Phi_k^*\nu^k$ in the remaining variables $P_1,\dots,P_k$:*

$$\int_{(P_{k+1},\dots,P_g)\in X^{g-k}} \Phi^*\nu^g(P_1,\dots,P_g) = \tfrac{g!(g-k)!}{k!}\Phi_k^*\nu^k(P_1,\dots,P_k).$$

*Proof.* By changing coordinates in $\mathbb{C}^g$ by a matrix $B$ with $B^2 = Y^{-1}$, we can restrict to the case, where $\nu$ is of the form $\nu = \frac{i}{2}\sum_{j=1}^g dZ_j \wedge d\bar{Z}_j$ and the pullbacks $\psi_j = I^*(dZ_j)$ form an orthonormal basis of $H^0(X,\Omega_X^1)$. Taking the $g$-th power of $\nu$ yields

$$\nu^g = \left(\tfrac{i}{2}\right)^g g! \cdot dZ_1 \wedge d\bar{Z}_1 \wedge \cdots \wedge dZ_g \wedge d\bar{Z}_g.$$

Since $\Phi^*(dZ_j) = \sum_{k=1}^g \psi_j(P_k)$, we get by pulling back $\nu^g$ with $\Phi$

$$\Phi^*\nu^g = \left(\tfrac{i}{2}\right)^g g! \sum_{\rho,\tau\in\mathrm{Sym}(g)} \bigwedge_{m=1}^g \psi_m(P_{\rho(m)}) \wedge \bar{\psi}_m(P_{\tau(m)}).$$

Since the $\psi_j$'s are orthonormal, only the summands with $\rho(j) = \tau(j)$ will remain after integrating over $P_j$. Hence, we can reduce to sum over permutations $\rho,\tau \in \mathrm{Sym}(k)$:

$$\begin{aligned}&\int_{(P_{k+1},\dots,P_g)\in X^{g-k}} \Phi^*\nu^g(P_1,\dots,P_g)\\ =&(g-k)!\left(\tfrac{i}{2}\right)^k g! \sum_{1\le j_1<\cdots<j_k\le g}\ \sum_{\rho,\tau\in\mathrm{Sym}(k)} \bigwedge_{m=1}^k \psi_{j_m}(P_{\rho(m)}) \wedge \bar{\psi}_{j_m}(P_{\tau(m)}),\end{aligned}$$

where the factor $(g-k)!$ comes from the permutations of the forms in $P_j$ for all $k < j \le g$. On the other hand, the pullback of

$$\nu^k = \left(\tfrac{i}{2}\right)^k k! \sum_{1\le j_1<\cdots<j_k\le g} dZ_{j_1} \wedge d\bar{Z}_{j_1} \wedge \cdots \wedge dZ_{j_k} \wedge d\bar{Z}_{j_k}$$

yields

$$\Phi^*\nu^k = \left(\tfrac{i}{2}\right)^k k! \sum_{1\le j_1<\cdots<j_k\le g}\ \sum_{\rho,\tau\in\mathrm{Sym}(k)} \bigwedge_{m=1}^k \psi_{j_m}(P_{\rho(m)}) \wedge \bar{\psi}_{j_m}(P_{\tau(m)}).$$

Now the lemma follows by comparing these formulas. □

Next, we calculate integrals of the Arakelov–Green function. As a consequence of Lemma 2.2.1, we get for all $k$

$$\tfrac{1}{(g!)^2}\int_{X^g} g(P_k,Q)\Phi^*\nu^g(P_1,\dots,P_g)=\int_X g(P_k,Q)\mu(P_k)=0.$$

For the terms $g(P_k,P_l)$ we get the following lemma relating their integrals to the Kawazumi–Zhang invariant $\varphi(X)$.

**Lemma 2.2.2.** *For $k\neq l$ we have*

$$\tfrac{1}{(g!)^2}\int_{X^g} g(P_k,P_l)\Phi^*\nu^g(P_1,\dots,P_g)=\tfrac{1}{2g(g-1)}\cdot\varphi(X).$$

*Proof.* As in the proof of Lemma 2.2.1, we change coordinates in $\mathbb{C}^g$, such that $\nu=\frac{i}{2}\sum_{j=1}^g dZ_j\wedge d\bar{Z}_j$ and the $\psi_j=I^*(dZ_j)$ form an orthonormal basis of $H^0(X,\Omega_X^1)$. We have $\nu^2=-\frac{1}{4}\sum_{p\neq q}dZ_p\wedge d\bar{Z}_p\wedge dZ_q\wedge d\bar{Z}_q$ and for the pullback by $\Phi_2$ we obtain $\Phi_2^*(dZ_j)=\psi_j(P_1)+\psi_j(P_2)$. Hence, we get for the pullback of $\nu^2$ after some calculations

$$\begin{aligned}\Phi_2^*\nu^2=\tfrac{1}{2}\sum_{p\neq q}(&\psi_p(P_1)\wedge\bar{\psi}_q(P_1)\wedge\psi_q(P_2)\wedge\bar{\psi}_p(P_2)\\ &-\psi_p(P_1)\wedge\bar{\psi}_p(P_1)\wedge\psi_q(P_2)\wedge\bar{\psi}_q(P_2)).\end{aligned}$$

Since $\mu=\frac{i}{2g}\sum_{j=1}^g\psi_j\wedge\bar{\psi}_j$, we get on the other hand

$$\mu(P_1)\mu(P_2)=-\tfrac{1}{4g^2}\sum_{p,q=1}^g\psi_p(P_1)\wedge\bar{\psi}_p(P_1)\wedge\psi_q(P_2)\wedge\bar{\psi}_q(P_2)$$

and by (1.2.7)

$$h_\Delta^2=2\mu(P_1)\mu(P_2)+\tfrac{1}{2}\sum_{p,q=1}^g\left(\psi_p(P_1)\wedge\bar{\psi}_q(P_1)\wedge\psi_q(P_2)\wedge\bar{\psi}_p(P_2)\right).$$

Putting this together, we obtain $h_\Delta^2=\Phi_2^*\nu^2-2(g^2-1)\mu(P_1)\mu(P_2)$. By (G3) in Section 1.2 the integral $\int_{X^2}g(P_1,P_2)\mu(P_1)\mu(P_2)$ vanishes. Using the defining equation (1.2.8) for $\varphi(X)$, we obtain

$$\tfrac{1}{2g(g-1)}\varphi(X)=\tfrac{1}{2g(g-1)}\int_{X^2}g(P_1,P_2)\Phi_2^*\nu^2=\tfrac{1}{(g!)^2}\int_{X^g}g(P_1,P_2)\Phi^*\nu^g,$$

where the latter equality is due to Lemma 2.2.1. Now the lemma follows by symmetry. □

The function $g(\sigma(P_1+\cdots+P_{g-1}), P_g)$ is defined on a dense subset of $X^g$. Hence, we can compute the integral over $X^g$ and we obtain the following relation.

**Lemma 2.2.3.** *It holds*

$$\frac{1}{(g!)^2}\int_{X^g} g(\sigma(P_1+\cdots+P_{g-1}), P_g)\Phi^*\nu^g = \tfrac{1}{2g}\varphi(X).$$

*Proof.* Denote by $X^{(g-1)}$ the $(g-1)$-th symmetric power of $X$. We have the canonical map

$$\widetilde{\Phi}_\Theta\colon X^{(g-1)} \to \Theta, \quad (P_1,\ldots,P_{g-1}) \to P_1+\cdots+P_{g-1}.$$

We denote $\widetilde{\Phi}_\Theta^{-1}(\Theta^{sm}) = \widetilde{X^{(g-1)}}$. The map $\widetilde{\Phi}_\Theta$ induces an isomorphism $\widetilde{X^{(g-1)}} \cong \Theta^{sm}$. In particular, we obtain the involution $\sigma$ also on $\widetilde{X^{(g-1)}}$. We define the map

$$\begin{gathered}\widetilde{\Phi}\colon \widetilde{X^{(g-1)}} \times X \to \mathrm{Pic}_{g-1}(X),\\ ((P_1,\ldots,P_{g-1}),P_g) \mapsto P_1+\cdots+P_g-Q\end{gathered}$$

and the map

$$\begin{gathered}\widetilde{\Phi}_\sigma\colon \widetilde{X^{(g-1)}} \times X \to \mathrm{Pic}_{g-1}(X),\\ ((P_1,\ldots,P_{g-1}),P_g) \mapsto \sigma(P_1+\cdots+P_{g-1})+P_g-Q.\end{gathered}$$

A direct computation as in the proof of Proposition 2.1.1 gives $\widetilde{\Phi}^*\nu^g = \widetilde{\Phi}_\sigma^*\nu^g$. Since $\widetilde{\Phi} = \widetilde{\Phi}_\sigma \circ (\sigma\times\mathrm{id}_X)$ and $(\sigma\times\mathrm{id}_X)$ is an automorphism, we can compute

$$\begin{aligned}&\frac{1}{(g!)^2}\int_{X^g} g(\sigma(P_1+\cdots+P_{g-1}), P_g)\Phi^*\nu^g\\ =&\frac{1}{g\cdot g!}\int_{\widetilde{X^{(g-1)}}\times X} g(\sigma(P_1+\cdots+P_{g-1}), P_g)\widetilde{\Phi}^*\nu^g\\ =&\frac{1}{g\cdot g!}\int_{\widetilde{X^{(g-1)}}\times X} g(P_1+\cdots+P_{g-1}, P_g)\widetilde{\Phi}_\sigma^*\nu^g\\ =&\frac{1}{g\cdot g!}\int_{\widetilde{X^{(g-1)}}\times X} g(P_1+\cdots+P_{g-1}, P_g)\widetilde{\Phi}^*\nu^g\\ =&\frac{1}{(g!)^2}\sum_{j=1}^{g-1}\int_{X^g} g(P_j, P_g)\Phi^*\nu^g.\end{aligned}$$

By Lemma 2.2.2 this equals $\frac{1}{2g}\varphi(X)$. □

# Chapter 3

# The hyperelliptic case

In this chapter we restrict to the case of hyperelliptic Riemann surfaces. In particular, we obtain an explicit description of the invariant $\delta$ in this case. Therefore, let $X$ be any hyperelliptic Riemann surface $X$ of genus $g \geq 2$ throughout this chapter.

## 3.1 Decomposition of theta functions

We give a decomposition of $\log \|\theta\|$ into a sum of Arakelov–Green functions and a certain invariant of $X$ and we state some consequences.

**Proposition 3.1.1.** *The function* $\log \|\theta\|$ *decomposes in the following way:*

$$\log \|\theta\|(P_1 + \cdots + P_g - Q) = S_g(X) + \sum_{j=1}^{g} g(P_j, Q) + \sum_{k<l} g(\sigma(P_k), P_l).$$

*Proof.* We consider

$$\alpha(P_1) = \log \|\theta\|(P_1 + \cdots + P_g - Q) - \sum_{j=1}^{g} g(P_j, Q) - \sum_{k<l} g(\sigma(P_k), P_l) \tag{3.1.1}$$

as a function in the variable $P_1$ by fixing the remaining points, such that each summand on the right hand side is well defined for at least some choices of $P_1$. For any point $P \in X$ the divisors $P + \sigma(P)$ and $2 \cdot \infty$ are linearly equivalent, see [Mum84, Chapter IIIa.§2.]. Hence, $P_1 + \cdots + P_g - Q$ is effective if $P_1 = \sigma(P_k)$ for some $k \neq 1$ or $P_1 = Q$. But $\|\theta\|(P_1 + \cdots + P_g - Q)$ has exactly $g$ zeros as a function in $P_1$, see [Mum83, Theorem II.3.1]. Therefore, $\alpha(P_1)$ has no poles. Further, we get

$$\partial\bar{\partial}\alpha(P_1) = \pi i I^* \nu(P_1) - g\pi i \mu(P_1) = 0$$

by ($\theta$2) and (G2) in Section 1.2. Hence, $\alpha(P_1)$ is a harmonic function on a compact space. Thus, it is constant. Analogously, we can show, that the expression (3.1.1) is constant as a function in $P_2, \dots, P_g$ or $Q$. Integrating with $\mu(P_1)\dots\mu(P_g)$ shows that $\alpha = S_g(X)$ since the Arakelov–Green functions vanish by (G3). □

As a corollary, we obtain a similar decomposition for the function $\log\|J\|$.

**Corollary 3.1.2.** *The function* $\log\|J\|$ *decomposes in the following way:*

$$\log\|J\|(P_1,\dots,P_g) = B(X) + \sum_{k<l} g(P_k,P_l) + (g-1)\sum_{k<l} g(P_k,\sigma(P_l)).$$

*Proof.* We apply the decomposition of $\log\|\theta\|$ in Proposition 3.1.1 to formula (1.2.4) and we eliminate $\delta(X)$ by (1.2.5). This gives the corollary. □

Another application of the decomposition in Proposition 3.1.1 is the following relation of invariants of $X$.

**Corollary 3.1.3.** *We have*

$$\log\|\varphi_g\|(X) = 4\tbinom{2g}{g-1}\left(\tfrac{g+1}{g}B(X) - (g-1)S_g(X) - (g+1)\log\pi\right).$$

*Proof.* Applying the decomposition of Proposition 3.1.1 to (1.3.3) gives:

$$\log\|\varphi_g\|(X) = 4\tbinom{2g+2}{g+1}S_g(X) + 4\tbinom{2g}{g-1}\sum_{1\le k<l\le 2g+2} g(W_k,W_l). \tag{3.1.2}$$

In the same way, the decomposition of Corollary 3.1.2 applied to (1.3.5) yields:

$$\begin{aligned}&\tbinom{2g+2}{g}g\log\pi + \tfrac{g+1}{4}\log\|\varphi_g\|(X) \\ =&\tbinom{2g+2}{g}B(X) + \tbinom{2g}{g-2}g\sum_{1\le k<l\le 2g+2} g(W_k,W_l).\end{aligned} \tag{3.1.3}$$

Now the lemma follows by combining (3.1.2) and (3.1.3). □

## 3.2 Comparison of integrals

In this section we prove the following relation between integrals of $\log\|\theta\|$.

**Theorem 3.2.1.** *It holds* $(g-1)H(X) = gS_g(X) - S_1(X)$.

The idea of the proof is to apply the decomposition in Proposition 3.1.1 to the two different expressions of $H(X)$ in (2.1.1) and (2.1.2). First, we prove the following two lemmas.

**Lemma 3.2.2.** *We have* $2S_1(X) = g(g-1)S_{g-1}(X) - (g+1)(g-2)S_g(X)$.

*Proof.* If we apply Proposition 3.1.1 to $\log\|\theta\|((g-k+1)P_1+P_2+\cdots+P_k-Q)$ and if we integrate with $\mu(P_1)\dots\mu(P_k)$, we get

$$S_k(X) = S_g(X) + \tfrac{(g-k)(g-k+1)}{2}\int_X g(\sigma(P),P)\mu(P). \tag{3.2.1}$$

If we do this for $k=1$ and for $k=g-1$, we can solve the two resulting equations for $S_1(X)$, $S_{g-1}(X)$ and $S_g(X)$. This yields the assertion of the lemma. □

The proof shows, that we can give similar relations for any three of the $S_j(X)$'s, but we will not need this.

**Lemma 3.2.3.** *For $k \neq l$ we have*

$$\tfrac{1}{(g!)^2}\int_{X^g} g(\sigma(P_k),P_l)\Phi^*\nu^g = \tfrac{1}{2g(g-1)}\cdot\varphi(X).$$

*Proof.* The involutions on $\mathrm{Pic}_{g-1}(X)$ and on $X$ are compatible in the sense that the divisors $\sigma(P_1+\cdots+P_{g-1})$ and $\sigma(P_1)+\cdots+\sigma(P_{g-1})$ are linearly equivalent. This follows, since $\sigma(P_j)+P_j$ and $2\cdot\infty$ are linearly equivalent and $(2g-2)\cdot\infty$ represents the canonical divisor class $K_X$, see [Mum84, Chapter IIIa §2.]. Thus, the lemma is a direct consequence of Lemma 2.2.3. □

*Proof of Theorem 3.2.1.* We can now prove the theorem using Lemma 2.2.2 and Lemma 3.2.3 to compute the terms which we get by applying the decomposition in Proposition 3.1.1 to the equations (2.1.1) and (2.1.2). This yields on the one hand

$$H(X) = \tfrac{1}{(g!)^2}\int_{X^g}\log\|\theta\|(P_1+\cdots+P_g-Q)\Phi^*\nu^g = S_g(X) + \tfrac{1}{4}\varphi(X),$$

and on the other hand

$$\begin{aligned} H(X) &= \tfrac{1}{(g!)^2}\int_{X^g}\log\|\theta\|(2P_1+P_2+\cdots+P_{g-1}-P_g)\Phi^*\nu^g \\ &= S_g(X) + \int_X g(\sigma(P),P)\mu(P) + \left(\tfrac{g(g+1)}{2}-1\right)\tfrac{1}{2g(g-1)}\varphi(X) \\ &= S_{g-1}(X) + \tfrac{(g+2)}{4g}\varphi(X). \end{aligned}$$

The last equality follows by (3.2.1). A simple computation yields

$$H(X) = \tfrac{g+2}{2} S_g(X) - \tfrac{g}{2} S_{g-1}(X).$$

Using Lemma 3.2.2, we can substitute $S_{g-1}(X)$ to obtain the formula in the theorem. □

As a corollary of the proof we get the following explicit expression for the Kawazumi–Zhang invariant.

**Corollary 3.2.4.** *It holds* $\varphi(X) = \frac{4}{g}(H(X) - S_1(X))$.

## 3.3 Explicit formulas for the delta invariant

Now we can deduce an explicit formula for $\delta(X)$. As mentioned in the introduction, Bost [Bos87, Proposition 4] stated the following expression for $\delta(X)$ for $g = 2$:

$$\delta(X) = -4H(X) - \tfrac{1}{4} \log \|\Delta_2\|(X) - 16 \log(2\pi).$$

We generalize this to hyperelliptic Riemann surfaces. Furthermore, we give a relation between $\delta(X)$ and $\varphi(X)$.

**Theorem 3.3.1.** *We have*

$$\delta(X) = -\tfrac{8(g-1)}{g} H(X) - \tbinom{2g}{g-1}^{-1} \log \|\Delta_g\|(X) - 8g \log 2\pi$$

*and*

$$\delta(X) = -24H(X) + 2\varphi(X) - 8g \log 2\pi.$$

*Proof.* First, we substitute $S_1(X)$ in formula (1.3.4) by the result of Theorem 3.2.1. This yields

$$\delta(X) = \tfrac{4(g-1)}{g} S_g(X) - \tfrac{4(g-1)^2}{g^2} H(X) - \tfrac{3g-1}{2gn} \log \|\Delta_g\|(X) - 8g \log 2\pi, \quad (3.3.1)$$

where we denote shortly $n = \binom{2g}{g-1}$. A combination of formula (1.2.5) and Corollary 3.1.3 yields

$$S_g(X) = \tfrac{g(g+1)}{g-1} \log 2\pi + \tfrac{g}{4n(g-1)} \log \|\Delta_g\|(X) + \tfrac{g+1}{8(g-1)} \delta(X). \quad (3.3.2)$$

If we now insert (3.3.2) for the $S_g(X)$-term in (3.3.1) and solve for $\delta(X)$, we obtain the first formula in the theorem. If we apply again equation (1.3.4) to this formula, we can eliminate the $\log \|\Delta_g\|(X)$-term to obtain

$$\delta(X) = -\tfrac{8(3g-1)}{g} H(X) - \tfrac{8}{g} S_1(X) - 8g \log 2\pi.$$

Now Corollary 3.2.4 gives the second formula in the theorem. □

For the applications to hyperelliptic curves over number fields in Section 6.2 we deduce the following formula for $\delta(X)$, which was also proved by de Jong in [dJo13, Corollary 1.8] by different methods.

**Corollary 3.3.2.** *It holds*

$$\delta(X) = -\tfrac{2(g-1)}{2g+1}\varphi(X) - \tfrac{3g}{(2g+1)}\tbinom{2g}{g-1}^{-1}\log\|\Delta_g\|(X) - 8g\log 2\pi.$$

*Proof.* This formula directly follows by combining the two formulas in Theorem 3.3.1. □

We can also conclude the following corollary about the Kawazumi–Zhang invariant $\varphi(X)$ and the modified discriminant $\|\Delta_g\|(X)$.

**Corollary 3.3.3.** *We obtain the following explicit formula for* $\varphi(X)$

$$\varphi(X) = \tfrac{4(2g+1)}{g}H(X) - \tfrac{1}{2}\tbinom{2g}{g-1}^{-1}\log\|\Delta_g\|(X).$$

*In particular, we get the upper bound* $\log\|\Delta_g\|(X) < -2(2g+1)\binom{2g}{g-1}\log 2$.

*Proof.* One gets the formula for $\varphi(X)$ by comparing the two formulas in Theorem 3.3.1 and solving for $\varphi(X)$, $\log\|\Delta_g\|(X)$ and $H(X)$. The bound follows by (1.2.9) and Proposition 1.1.1. □

Von Känel has given an upper bound for $\|\Delta_g\|(X)$ in [vKä14a, Lemma 5.4] by bounding the function $\|\theta\|$ similarly as we will do in the proof of Lemma 4.5.1. However, our bound for $\|\Delta_g\|(X)$ is much sharper. In particular, it decreases for growing $g$.

**Example 3.3.4.** *The formulas in Theorem 3.3.1 and Corollary 3.3.3 allows us to compute the invariants $\delta$ and $\varphi$ effectively for hyperelliptic Riemann surfaces. For any integer $n \geq 5$ consider the hyperelliptic Riemann surface $X_n$ given by the projective closure of the complex, affine curve defined by*

$$y^2 = x^n + a,$$

*where $a \in \mathbb{C} \setminus \{0\}$. The isomorphism class of $X_n$ does not depend on $a$, as one sees by a change of coordinates. It is also isomorphic to the hyperelliptic Riemann surface associated to the equation $y^2 + y = x^n$. Using the software Mathematica we obtain the following values:*

| $n$ | *Genus of* $X_n$ | $\log\|\Delta_g\|(X_n)$ | $H(X_n)$ | $\delta(X_n)$ | $\varphi(X_n)$ |
|---|---|---|---|---|---|
| 5 | 2 | $-43.14$ | $-0.485$ ($\pm 0.003$) | $-16.68$ | 0.54 |
| 6 | 2 | $-44.34$ | $-0.495$ ($\pm 0.001$) | $-16.34$ | 0.59 |
| 7 | 3 | $-239.75$ | $-0.706$ ($\pm 0.019$) | $-24.36$ | 1.40 |
| 8 | 3 | $-246.58$ | $-0.719$ ($\pm 0.011$) | $-23.84$ | 1.51 |

*The values of $H(X_5)$, $\|\Delta_2\|(X_5)$ and $\delta(X_5)$ were also computed in [BMM90]. More recently, Pioline found in [Pio15] formulas for the invariants $\delta$ and $\varphi$ of Riemann surfaces of genus 2, which allow a noticeably more efficient computation of $\delta$ and $\varphi$ than our formulas. In particular, he computed the values of $\delta(X_5)$, $\varphi(X_5)$, $\delta(X_6)$ and $\varphi(X_6)$ in [Pio15, Section 4.1].*

The invariant $\|\Delta_g\|(X)$ can be computed much more efficiently than the invariant $H(X)$. Moreover, the Noether formula predicts, that $\delta$ is the archimedean analogue of the logarithm of the discriminant of the finite places. Indeed, $\delta$ is essentially the logarithm of the norm of the modular discriminant for elliptic Riemann surfaces. Hence, it may be interesting to approximate $\delta(X)$ by $\log\|\Delta_g\|(X)$ for hyperelliptic Riemann surfaces.

**Corollary 3.3.5.** *We have the following relation between the invariants $\delta(X)$ and $\|\Delta_g\|(X)$:*

$$-\tfrac{1}{n}\log\|\Delta_g\|(X)+2(g-1)\log 2<\delta(X)+8g\log 2\pi<\tfrac{-3g}{(2g+1)n}\log\|\Delta_g\|(X),$$

*where we write shortly $n=\binom{2g}{g-1}$.*

*Proof.* The first bound directly follows from the first formula in Theorem 3.3.1 and the bound in Proposition 1.1.1. The second inequality follows by applying the bound $\varphi(X)>0$ to the formula in Corollary 3.3.2. □

## 3.4 A generalized Rosenhain formula

Finally, we apply the decomposition in Proposition 3.1.1 to give an absolute value answer to a conjecture by Guàrdia in [Guà02, Conjecture 14.1]. Rosenhain stated in [Ros51] an identity for the case $g=2$, which can be written in our setting as

$$\|J\|(W,W')=\pi^2\prod_{W''\neq W,W'}\|\theta\|(W''+W-W'),$$

where $W,W'$ are two different Weierstraß points and the product runs over all Weierstraß points $W''$ different from $W$ and $W'$. Looking for a generalization to genus $g\geq 2$, de Jong has found formula (1.3.5). We deduce the following more general result.

**Theorem 3.4.1.** *For any permutation $\tau\in\mathrm{Sym}(2g+2)$ it holds*

$$\|J\|(W_{\tau(1)},\dots,W_{\tau(g)})=\pi^g\prod_{j=g+1}^{2g+2}\|\theta\|(W_{\tau(1)}+\dots+W_{\tau(g)}-W_{\tau(j)}).$$

*Proof.* First, we compare the applications of the decomposition in Proposition 3.1.1 to (1.3.2) and (1.3.3). This yields

$$8\binom{2g-1}{g-1}\sum_{1\le k<l\le 2g+1} g(W_k,W_l)=4\binom{2g}{g-1}\sum_{1\le k<l\le 2g+2} g(W_k,W_l).$$

An elementary calculation gives

$$\sum_{1\le k<l\le 2g+1} g(W_k,W_l)=g\sum_{k=1}^{2g+1} g(W_k,W_{2g+2}).$$

The decomposition corresponding to (1.3.2) is

$$\log\|\varphi_g\|(X)=8\binom{2g+1}{g+1}S_g(X)+8\binom{2g-1}{g-1}\sum_{1\le k<l\le 2g+1} g(W_k,W_l).$$

Hence, we get

$$8g\binom{2g-1}{g-1}\sum_{k=1}^{2g+1} g(W_k,W_{2g+2})=\log\|\varphi_g\|(X)-8\binom{2g+1}{g+1}S_g(X).$$

Since this does not depend on the choice of the Weierstraß point at infinity, we more generally get for a fixed $1\le m\le 2g+2$

$$8g\binom{2g-1}{g-1}\sum_{\substack{k=1\\k\ne m}}^{2g+2} g(W_k,W_m)=\log\|\varphi_g\|(X)-8\binom{2g+1}{g+1}S_g(X).$$

Summing this for $m=\tau(1),\ldots,\tau(g)$ and using Corollary 3.1.3 to eliminate the term $\log\|\varphi_g\|(X)$, we get

$$\sum_{j=1}^{g}\sum_{\substack{k=1\\k\ne\tau(j)}}^{2g+2} g(W_k,W_{\tau(j)})=B(X)-(g+2)S_g(X)-g\log\pi.$$

Now we can conclude the theorem by the following calculation:

$$\begin{aligned}
&\sum_{j=g+1}^{2g+2} \log \|\theta\|(W_{\tau(1)} + \cdots + W_{\tau(g)} - W_{\tau(j)}) \\
=&(g+2)S_g(X) + (g+2) \sum_{1\le k<l\le g} g(W_{\tau(k)}, W_{\tau(l)}) + \sum_{j=g+1}^{2g+2} \sum_{k=1}^{g} g(W_{\tau(k)}, W_{\tau(j)}) \\
=&(g+2)S_g(X) + g \sum_{1\le k<l\le g} g(W_{\tau(k)}, W_{\tau(l)}) + \sum_{j=1}^{g} \sum_{\substack{k=1 \\ k\neq\tau(j)}}^{2g+2} g(W_k, W_{\tau(j)}) \\
=&(g+2)S_g(X) + g \sum_{1\le k<l\le g} g(W_{\tau(k)}, W_{\tau(l)}) + B(X) - (g+2)S_g(X) - g\log\pi \\
=&\log \|J\|(W_{\tau(1)}, \ldots, W_{\tau(g)}) - g\log\pi.
\end{aligned}$$

This completes the proof. □

# Chapter 4

# The general case

We prove our main result in this chapter, see Theorem 4.4.1, and we deduce some applications, for example a lower bound for $\delta$ and an explicit expression and an upper bound for the Arakelov–Green function.

## 4.1 Forms on universal families

In this section we discuss canonical forms on the universal family of compact and connected Riemann surfaces of fixed genus and on the universal family of principally polarised complex abelian varieties of fixed dimension with level 2 structure. We use these forms to compute the application of $\partial\overline{\partial}$ to invariants of Riemann surfaces considered as functions on the moduli space.

Let $g \geq 3$. Denote by $\mathcal{M}_g$ the moduli space of compact and connected Riemann surfaces of genus $g$ and by $q\colon \mathcal{C}_g \to \mathcal{M}_g$ the universal family of compact and connected Riemann surfaces of genus $g$. The Arakelov–Green function defines a function $G\colon \mathcal{C}_g \times_{\mathcal{M}_g} \mathcal{C}_g \to \mathbb{R}_{\geq 0}$, which again defines a metric on $\mathcal{O}(\Delta)$, where $\Delta \subseteq \mathcal{C}_g \times_{\mathcal{M}_g} \mathcal{C}_g$ is the diagonal. This induces a metric on the relative tangent bundle $T_{\mathcal{C}_g/\mathcal{M}_g}$, since $T_{\mathcal{C}_g/\mathcal{M}_g}$ is the normal bundle of $\Delta$. Denote by $h = c_1(\mathcal{O}(\Delta))$ the first Chern form of $\mathcal{O}(\Delta)$, that means, we have an equality

$$\tfrac{1}{\pi i}\partial\overline{\partial}\log G = h - \delta_\Delta$$

of currents on $\mathcal{C}_g \times_{\mathcal{M}_g} \mathcal{C}_g$. Further, we set $e^A = h|_\Delta$, which is the first Chern form of $T_{\mathcal{C}_g/\mathcal{M}_g}$. We write $e_1^A = \int_q (e^A)^2$. A direct calculation, see also [dJo14b, Proposition 5.3], gives the equality

$$\tfrac{1}{\pi i}\partial\overline{\partial}\varphi = \int_{q_2} h^3 - e_1^A \tag{4.1.1}$$

of forms on $\mathcal{M}_g$, where $q_2\colon \mathcal{C}_g \times_{\mathcal{M}_g} \mathcal{C}_g \to \mathcal{M}_g$ is the canonical morphism.

Denote by $\det q_*\Omega^1_{\mathcal{C}_g/\mathcal{M}_g}$ the determinant of the Hodge bundle of $\mathcal{C}_g$ over $\mathcal{M}_g$ equipped with the metric induced by (1.2.3) and write $\omega_{\mathrm{Hdg}}$ for its first Chern form. The invariant $\delta$ satisfies

$$\tfrac{1}{\pi i}\partial\overline{\partial}\delta = e_1^A - 12\omega_{\mathrm{Hdg}}, \tag{4.1.2}$$

see for example [dJo14b, Section 10].

Now we consider $\mathcal{M}_g[2]$, the moduli space of compact and connected Riemann surfaces of genus $g$ with level 2 structure, see for example [HL97, Section 7.4] for a precise definition. Denote $\pi\colon \mathcal{X}_g \to \mathcal{M}_g[2]$ for the universal compact and connected Riemann surface over $\mathcal{M}_g[2]$. We will fix some notation. We write $\mathcal{X}_g^n$ for the product $\mathcal{X}_g \times_{\mathcal{M}_g[2]} \cdots \times_{\mathcal{M}_g[2]} \mathcal{X}_g$ with $n$ factors over $\mathcal{M}_g[2]$ and $\pi_n\colon \mathcal{X}_g^n \to \mathcal{M}_g[2]$ for the canonical morphism. Further, we denote $\mathcal{X}_g^{(n)}$ for the corresponding symmetric product and $\rho_n\colon \mathcal{X}^n \to \mathcal{X}^{(n)}$ for the canonical map. For any $m, n$ with $m \leq n$ and pairwise different $j_1, \dots, j_m$ we denote by $pr_{j_1,\dots,j_m}\mathcal{X}_g^n \to \mathcal{X}_g^m$ the projection to the $j_1$-th, $\dots$, $j_m$-th factors. Moreover, we write $pr^{j_1,\dots,j_m}\colon \mathcal{X}_g^n \to \mathcal{X}_g^{n-m}$ for the projection forgetting the $j_1$-th, $\dots$, $j_m$-th factors. We obtain forms $h$ on $\mathcal{X}_g^2$, $e^A$ on $\mathcal{X}_g$ and $e_1^A$ and $\omega_{\mathrm{Hdg}}$ on $\mathcal{M}_g[2]$ by pulling back the forms $h$, $e^A$, $e_1^A$ and $\omega_{\mathrm{Hdg}}$ defined above by the maps forgetting the level 2 structure.

Further, we denote by $\mathcal{A}_g[2]$ the moduli space of principally polarised complex abelian varieties with level 2 structure and we write $p\colon \mathcal{U}_g \to \mathcal{A}_g[2]$ for the universal principally polarised complex abelian variety over $\mathcal{A}_g[2]$. There exists a 2-form $\omega_0$ on $\mathcal{U}_g$ such that the restriction of $\omega_0$ to a principally polarised abelian variety $(A, \Theta)$ with arbitrary level 2 structure considered as a fibre of $p$ is $\nu_{(A,\Theta)}$ and the restriction of $\omega_0$ along the zero section of $p$ is trivial, see for example [HR01]. Without risk of confusions, we also write $\omega_{\mathrm{Hdg}}$ for the first Chern form of $\det p_*\Omega^1_{\mathcal{U}_g/\mathcal{A}_g[2]}$ endowed with its $L^2$-metric. If we denote the Torelli map by $t\colon \mathcal{M}_g[2] \to \mathcal{A}_g[2]$, it holds $t^*\omega_{\mathrm{Hdg}} = \omega_{\mathrm{Hdg}}$ as forms on $\mathcal{M}_g[2]$, see [Szp85b, Lemme 3.2.1].

Next, we would like to define the function $\|\theta\|$ on $\mathcal{U}_g$. However, there is no canonical theta divisor for an arbitrary principally polarised complex abelian variety. But for any compact and connected Riemann surface $X$, there is a canonical theta divisor in $\mathrm{Pic}_{g-1}(X)$ given by the image of the canonical map $X^{(g-1)} \to \mathrm{Pic}_{g-1}(X)$. Every theta characteristic $\alpha$ of $X$ defines a theta divisor $\Theta_\alpha \subseteq \mathrm{Jac}(X)$, see (1.2.1). On $\mathcal{M}_g[2]$ we can consistently choose a theta characteristic on each curve. Hence, we obtain a theta characteristic $\alpha$ of $\mathcal{X}_g$. For every such theta characteristic $\alpha$ of $\mathcal{X}_g$ we get a theta divisor $\Theta_\alpha$ in $\mathcal{U}_g$. Using the properties of uniqueness $(\theta 1)$–$(\theta 3)$ in Section 1.1 on each fibre of $p$, we obtain a function $\|\theta_\alpha\|$ on $\mathcal{U}_g$. We define a metric on the line bundle $\mathcal{O}(\Theta_\alpha)$ on $\mathcal{U}_g$ by $\|\theta_\alpha\|$. This line bundle has first Chern form $\omega_0 + \frac{1}{2}\omega_{\mathrm{Hdg}}$, see

[HR01, Proposition 2]. Hence, we obtain

$$\tfrac{1}{\pi i}\partial\overline{\partial}\log\|\theta_\alpha\| = \omega_0 + \tfrac{1}{2}\omega_{\mathrm{Hdg}} - \delta_{\Theta_\alpha}.$$

We would like to express $\frac{1}{\pi i}\partial\overline{\partial}H(X)$ by the forms $e_1^A$, $\int_{\pi_2} h^3$ and $\omega_{\mathrm{Hdg}}$. For this purpose, we fix a theta characteristic $\alpha$ of $\mathcal{X}_g$ and we consider the map

$$\begin{aligned}\gamma' \colon \mathcal{X}_g^{(g-1)} \times_{\mathcal{M}_g[2]} \mathcal{X}_g^2 &\to \mathcal{U}_g,\\ [X;(P_1,\ldots,P_{g-1}),P_g,P_{g+1}] &\mapsto [\mathrm{Jac}(X);P_1+\cdots+P_g-P_{g+1}-\alpha].\end{aligned}$$

Further, we write $\gamma = \gamma' \circ (\rho_{g-1}\times \mathrm{id}_{\mathcal{X}_g^2})\colon \mathcal{X}_g^{g+1} \to \mathcal{U}_g$. Note that $\gamma'$ and $\gamma$ depend on the choice of $\alpha$. The restriction of $\gamma$ to a fibre of $\pi_{g+1}$, that means to the $(g+1)$-th power of a compact and connected Riemann surface $X$ with a level 2 structure inducing a theta characteristic $\alpha_X$, is

$$\gamma|_{X^{g+1}} \colon X^{g+1} \to \mathrm{Jac}(X), \quad (P_1,\ldots,P_{g+1}) \mapsto P_1+\cdots+P_g-P_{g+1}-\alpha_X.$$

Fixing a Riemann surface $X\in\mathcal{M}_g$ and a point $Q\in X$ we obtain a map

$$s_Q\colon X^g \to X^{g+1}, \quad (P_1,\ldots,P_g)\mapsto (P_1,\ldots,P_g,Q),$$

which is a section of $pr^{g+1}|_{X^{g+1}}\colon X^{g+1}\to X^g$. As shown in Section 2.1, we have

$$H(X) = \tfrac{1}{(g!)^2}\int_{X^g} \log\|\theta_{\alpha_X}\|(P_1+\cdots+P_g-Q-\alpha_X)((\gamma|_{X^{g+1}})\circ s_Q)^*\nu^g.$$

A direct computation yields

$$\begin{aligned}&\tfrac{1}{(g!)^2}\int_{X^g} \log\|\theta_{\alpha_X}\|(P_1+\cdots+P_g-Q-\alpha_X)((\gamma|_{X^{g+1}})\circ s_Q)^*\nu^g\\ =&\tfrac{1}{(g!)^2}\int_{pr_{g+1}|_{X^{g+1}}} \log\|\theta_{\alpha_X}\|(P_1+\cdots+P_g-P_{g+1}-\alpha_X)(\gamma|_{X^{g+1}})^*\nu^g,\end{aligned}$$

which shows that the latter equals $H(X)$ and it is independent of the choice of the point $P_{g+1}$.

The restriction of $\omega_{\mathrm{Hdg}}$ to a fibre of $p$ is trivial and the restriction of $\omega_0$ to a fibre of $p$ equals $\nu$. Hence, we obtain

$$H(X) = \tfrac{1}{(g!)^2}\int_{pr_{g+1}} \log\|\theta_\alpha\|(P_1+\cdots+P_g-P_{g+1}-\alpha)\gamma^*(\omega_0+\tfrac{1}{2}\omega_{\mathrm{Hdg}})^g.$$

Using this expression, we compute $\frac{1}{\pi i}\partial\overline{\partial}H(X)$ by applying the Laplace operator $\frac{1}{\pi i}\partial\overline{\partial}$ on $\mathcal{X}_g^{g+1}$:

$$\begin{aligned}
&\tfrac{1}{\pi i}\partial\overline{\partial}\int_{pr_{g+1}} \log\|\theta_\alpha\|(P_1+\cdots+P_g-P_{g+1}-\alpha)\gamma^*(\omega_0+\tfrac{1}{2}\omega_{\mathrm{Hdg}})^g\\
&=\int_{pr_{g+1}}\gamma^*(\omega_0+\tfrac{1}{2}\omega_{\mathrm{Hdg}})^{g+1}-\int_{pr_{g+1}}\gamma^*\left(\delta_{\Theta_\alpha}\right)\gamma^*(\omega_0+\tfrac{1}{2}\omega_{\mathrm{Hdg}})^g\\
&=\int_{pr_{g+1}}\gamma^*\omega_0^{g+1}+\tfrac{g+1}{2}\int_{pr_{g+1}}\gamma^*\omega_0^g\wedge\omega_{\mathrm{Hdg}}-\int_{pr_{g+1}}\gamma^*\left(\delta_{\Theta_\alpha}\right)\gamma^*\omega_0^g\\
&\quad-\tfrac{g}{2}\int_{pr_{g+1}}\gamma^*(\delta_{\Theta_\alpha})\gamma^*\omega_0^{g-1}\wedge\omega_{\mathrm{Hdg}}.
\end{aligned}$$

Since the restriction of $\omega_{\mathrm{Hdg}}$ to a fibre of $pr_{g+1}$ is trivial and it holds $\int_A \nu^g_{(A,\Theta)}=\int_\Theta \nu^{g-1}_{(A,\Theta)}=g!$ for any principally polarised complex abelian variety $(A,\Theta)$, we get

$$\tfrac{g+1}{2}\int_{pr_{g+1}}\gamma^*\omega_0^g\wedge\omega_{\mathrm{Hdg}}=\tfrac{(g+1)\cdot(g!)^2}{2}\omega_{\mathrm{Hdg}}\quad\text{and}$$

$$\tfrac{g}{2}\int_{pr_{g+1}}\gamma^*\left(\delta_{\Theta_\alpha}\right)\gamma^*\omega_0^{g-1}\wedge\omega_{\mathrm{Hdg}}=\tfrac{g\cdot(g!)^2}{2}\omega_{\mathrm{Hdg}}.$$

Therefore, we obtain

$$\tfrac{1}{\pi i}\partial\overline{\partial}H(X)=\tfrac{1}{2}\omega_{\mathrm{Hdg}}+\tfrac{1}{(g!)^2}\left(\int_{pr_{g+1}}\gamma^*\omega_0^{g+1}-\int_{pr_{g+1}}\gamma^*\left(\delta_{\Theta_\alpha}\right)\gamma^*\omega_0^g\right).\qquad(4.1.3)$$

Thus, we have to compute the form $\gamma^*\omega_0$.

## 4.2 Deligne pairings

In this section we introduce the Deligne pairing for hermitian line bundles as it was defined by Deligne in [Del85, Section 6] and extended to arbitrary relative dimension by Zhang in [Zha96]. We will use it to study the form $\gamma^*\omega_0$.

Let $q\colon \mathfrak{X}\to S$ be a smooth, flat and projective morphism of complex manifolds of pure relative dimension $n$, and let $\mathcal{L}_0,\dots,\mathcal{L}_n$ be hermitian line bundles on $\mathfrak{X}$. Then the line bundle $\langle\mathcal{L}_0,\dots,\mathcal{L}_n\rangle(\mathfrak{X}/S)$ is the line bundle on $S$, which is locally generated by symbols $\langle l_0,\dots,l_n\rangle$, where the $l_j's$ are sections of the respective $\mathcal{L}_j's$ such that their divisors have no intersection, and if for some $0\le j\le n$ and some function $f$ on $\mathfrak{X}$ the intersection

$\prod_{k\neq j} \operatorname{div}(l_k) = \sum_i n_i Y_i$ is finite over $S$ and it has empty intersection with $\operatorname{div}(f)$, then it holds the relation

$$\langle l_0, \dots, l_{j-1}, f \cdot l_j, l_{j+1}, \dots, l_n \rangle = \prod_i \mathrm{Norm}_{Y_i/S}(f)^{n_i} \langle l_0, \dots, l_n \rangle.$$

By induction we define a metric on $\langle \mathcal{L}_0, \dots, \mathcal{L}_n \rangle(\mathfrak{X}/S)$ such that

$$\log \|\langle l_0, \dots, l_n \rangle\| = \log \|\langle l_0, \dots, l_{n-1} \rangle\|(\operatorname{div}(l_n)) + \int_q \log \|l_n\| \bigwedge_{i=0}^{n-1} c_1(\mathcal{L}_i),$$

where $c_1(\mathcal{L})$ denotes the first Chern form of a hermitian line bundle $\mathcal{L}$.

In the following, we list some properties, which can be found in [Zha96, Section 1]. The Deligne pairing is multilinear and symmetric and it satisfies

$$c_1(\langle \mathcal{L}_0, \dots, \mathcal{L}_n \rangle) = \int_q \bigwedge_{i=0}^{n} c_1(\mathcal{L}_i). \tag{4.2.1}$$

Further, let $\phi\colon \mathfrak{X} \to \mathcal{Y}$ be a smooth, flat and projective morphism of complex manifolds over $S$ with $m_1 = \dim \mathcal{Y}/S$ and $m_2 = \dim \mathfrak{X}/\mathcal{Y}$, $\mathcal{K}_0, \dots, \mathcal{K}_{m_2}$ hermitian line bundles on $\mathfrak{X}$ and $\mathcal{L}_1, \dots, \mathcal{L}_{m_1}$ hermitian line bundles on $\mathcal{Y}$. We have an isometry

$$\begin{aligned} &\langle \mathcal{K}_0, \dots, \mathcal{K}_{m_2}, \phi^*\mathcal{L}_1, \dots, \phi^*\mathcal{L}_{m_1} \rangle(\mathfrak{X}/S) \\ \cong &\langle \langle \mathcal{K}_0, \dots, \mathcal{K}_{m_2} \rangle(\mathfrak{X}/\mathcal{Y}), \mathcal{L}_1, \dots, \mathcal{L}_{m_1} \rangle(\mathcal{Y}/S). \end{aligned} \tag{4.2.2}$$

If $m_2 = 1$ and $\mathcal{K}_0 = \phi^*\mathcal{L}_0$ for some hermitian line bundle $\mathcal{L}_0$ on $\mathcal{Y}$, we obtain

$$c_1(\langle \mathcal{K}_1, \phi^*\mathcal{L}_0, \dots, \phi^*\mathcal{L}_{n-1} \rangle(\mathfrak{X}/S)) = \deg(\mathcal{K}_1) \cdot c_1(\langle \mathcal{L}_0, \dots, \mathcal{L}_{n-1} \rangle(\mathcal{Y}/S)). \tag{4.2.3}$$

Moreover, for general $m_2$ and hermitian line bundles $\mathcal{L}_0, \dots, \mathcal{L}_{m_1+1}$ on $\mathcal{Y}$, we have the isometry

$$\langle \mathcal{K}_1, \dots \mathcal{K}_{m_2-1}, \phi^*\mathcal{L}_0, \dots, \phi^*\mathcal{L}_{m_1+1} \rangle(\mathfrak{X}/S) = \mathcal{O}_S. \tag{4.2.4}$$

We will often omit $(\mathfrak{X}/S)$ in the notation and we will also use the shorter notation $\mathcal{L}_0^{\langle n+1 \rangle} = \langle \mathcal{L}_0, \dots, \mathcal{L}_0 \rangle$, where the $\mathcal{L}_0$ occurs $(n+1)$ times on the right hand side.

We apply this to the family $pr^{g+2}\colon \mathcal{X}_g^{g+2} \to \mathcal{X}_g^{g+1}$. For any positive integers $j \leq k$ we have a canonical section of $pr^{k+1}$:

$$s_{k+1,j}\colon \mathcal{X}_g^k \to \mathcal{X}_g^{k+1}, \quad [X; P_1, \dots, P_k] \mapsto [X; P_1, \dots, P_k, P_j].$$

We set $\mathcal{L}=\mathcal{O}(2s_{g+2,1}+\cdots+2s_{g+2,g}-2s_{g+2,g+1})\otimes pr_{g+2}^*T$ as a line bundle on $\mathcal{X}_g^{g+2}$, where we write $T=T_{\mathcal{X}_g/\mathcal{M}_g[2]}$ for the relative tangent bundle. The first Chern form of $\mathcal{L}$ vanishes if we restrict to any fibre of $pr^{g+2}\colon \mathcal{X}_g^{g+2}\to\mathcal{X}_g^{g+1}$. In particular, $\mathcal{L}$ is of degree 0 on each fibre of $pr^{g+2}$, such that we obtain a section of the Jacobian bundle $\mathcal{U}_g\times_{\mathcal{A}_g[2]}\mathcal{X}_g^{g+1}\to\mathcal{X}_g^{g+1}$. By definition this section equals $([2]\circ\gamma)\times\mathrm{id}_{\mathcal{X}_g^{g+1}}$, where $[2]$ denotes the multiplication with 2 on $\mathcal{U}_g$. By a result due to de Jong [dJo14b, Proposition 6.3], we have $c_1(\mathcal{L}^{\langle 2\rangle})=-2([2]\circ\gamma)^*\omega_0$. Thus, we can compute

$$\gamma^*\omega_0=\tfrac{1}{4}([2]\circ\gamma)^*\omega_0=-\tfrac{1}{8}c_1(\mathcal{L}^{\langle 2\rangle}). \tag{4.2.5}$$

If $s$ is a section of $pr^{g+2}$ and $\mathcal{L}_0$ any hermitian line bundle on $\mathcal{X}_g^{g+2}$, we obtain a canonical isometry $\langle\mathcal{O}(s),\mathcal{L}_0\rangle\cong s^*\mathcal{L}_0$. Hence, it follows

$$\langle\mathcal{O}(s_{g+2,j}),\mathcal{O}(s_{g+2,j})\rangle\cong s_{g+2,j}^*\mathcal{O}(s_{g+2,j})\cong pr_j^*T, \tag{4.2.6}$$

where the last isometry follows, since $s_{g+2,j}^*\mathcal{O}(s_{g+2,j})$ is the pullback of the line bundle $s_{2,1}^*\mathcal{O}(\Delta)$ by the projection $pr_j\colon\mathcal{X}_g^{g+1}\to\mathcal{X}_g$ to the $j$-th factor and $s_{2,1}$ is the diagonal embedding $\mathcal{X}_g\to\mathcal{X}_g^2$, such that $s_{2,1}^*\mathcal{O}(\Delta)\cong T$. Moreover, we have for $j\neq k$

$$\langle\mathcal{O}(s_{g+2,j}),\mathcal{O}(s_{g+2,k})\rangle\cong s_{g+2,j}^*\mathcal{O}(s_{g+2,k})\cong pr_{j,k}^*\mathcal{O}(\Delta) \tag{4.2.7}$$

and for all $1\leq j\leq g+1$

$$\langle\mathcal{O}(s_{g+2,j}),pr_{g+2}^*T\rangle=s_{g+2,j}^*pr_{g+2}^*T=pr_j^*T.$$

Now we can express the line bundle $\mathcal{L}^{\langle 2\rangle}$ by

$$\begin{aligned}\mathcal{L}^{\langle 2\rangle}\cong&\left(\bigotimes_{j=1}^g pr_j^*T\otimes\bigotimes_{j=1}^g pr_{j,g+1}^*\mathcal{O}(\Delta)^\vee\otimes\bigotimes_{j<k}^g pr_{j,k}^*\mathcal{O}(\Delta)\right)^{\otimes 8}\\&\otimes\left(pr_{g+2}^*T\right)^{\langle 2\rangle},\end{aligned} \tag{4.2.8}$$

where we denote $\mathscr{L}^\vee$ for the dual of a line bundle $\mathscr{L}$. We define $\widetilde{\mathcal{L}}$ by

$$\widetilde{\mathcal{L}}=\bigotimes_{j=1}^g pr_j^*T\otimes\bigotimes_{j=1}^g pr_{j,g+1}^*\mathcal{O}(\Delta)^\vee\otimes\bigotimes_{j<k}^g pr_{j,k}^*\mathcal{O}(\Delta),$$

such that $\mathcal{L}^{\langle 2\rangle}=\widetilde{\mathcal{L}}^{\otimes 8}\otimes\left(pr_{g+2}^*T\right)^{\langle 2\rangle}$. It holds $c_1(pr_{g+2}^*T)=pr_{g+2}^*e^A$ and hence, we deduce by (4.2.1) that $c_1\left(\left(pr_{g+2}^*T\right)^{\langle 2\rangle}\right)=e_1^A$. Since the restriction of $e_1^A$

to a fibre of $\pi_{g+1}$ is trivial and further, the restriction of $c_1(\widetilde{\mathcal{L}})$ to a fibre $X^{g+1}$ of $\pi_{g+1}$ is equal to $-(\gamma^*\omega_0)|_{X^{g+1}} = -(\gamma|_{X^{g+1}})^*\nu_{\mathrm{Jac}(X)}$, we get by (4.2.5)

$$\begin{aligned}\int_{pr_{g+1}} \gamma^*\omega_0^{g+1} &= \left(-\tfrac{1}{8}\right)^{g+1} \int_{pr_{g+1}} c_1(\mathcal{L}^{\langle 2\rangle})^{g+1} \\ &= (-1)^{g+1} \cdot \left( \int_{pr_{g+1}} c_1(\widetilde{\mathcal{L}})^{g+1} + \tfrac{g+1}{8} \int_{pr_{g+1}} c_1(\widetilde{\mathcal{L}})^{g} \wedge e_1^A \right) \\ &= (-1)^{g+1} \cdot \int_{pr_{g+1}} c_1(\widetilde{\mathcal{L}})^{g+1} - \tfrac{g+1}{8} \cdot (g!)^2 e_1^A.\end{aligned}$$

Next, we compute the second integral in equation (4.1.3). Denote by $\mathcal{H}_g[2]$ the moduli space of hyperelliptic Riemann surfaces of genus $g$ with level 2 structure. We restrict for the rest of this section to the open subspace $\mathcal{M}'_g = \mathcal{M}_g[2] \setminus \mathcal{H}_g[2]$ of $\mathcal{M}_g[2]$. In particular, we sloppily write $\mathcal{X}_g$ for the restriction $\mathcal{X}_g \times_{\mathcal{M}_g[2]} \mathcal{M}'_g$, $\gamma$ instead of $\gamma|_{\mathcal{X}_g \times_{\mathcal{M}_g[2]} \mathcal{M}'_g}$, etc. The singular locus $\Theta_\alpha^{sing}$ of the divisor $\Theta_\alpha$ has codimension 4 in the restriction of $\mathcal{U}_g/\mathcal{A}_g[2]$ to $\mathcal{M}'_g$, and its preimage under the map

$$\gamma_{\Theta_\alpha} \colon \mathcal{X}_g^{g-1} \to \Theta_\alpha \quad [X; P_1, \ldots, P_{g-1}] \mapsto [\mathrm{Jac}(X); P_1 + \cdots + P_{g-1} - \alpha]$$

has codimension 2. This follows for example from the proof of [BL04, Proposition 11.2.8]. Hence, the points $[X; P_1 + \cdots + P_g - P_{g+1}] \in pr_{g+1}^{-1}([X; P_{g+1}])$ with the property $P_1 + \cdots + P_{g-1} \in \Theta_\alpha^{sing}$ form a subspace of the fibre $pr_{g+1}^{-1}([X; P_{g+1}])$ of dimension at most $(g-2)$. Since the current $\gamma^*(\delta_{\Theta_\alpha})$ restricts the space for the integration to a space of dimension $g-1$, it is enough to integrate over the subspace where $P_1 + \cdots + P_{g-1} \notin \Theta_\alpha^{sing}$. Write $\widetilde{\mathcal{X}_g^{(g-1)}}$ for the subspace of $\mathcal{X}_g^{(g-1)}$, where $P_1 + \cdots + P_{g-1} \notin \Theta_\alpha^{sing}$. The canonical involution on the universal Jacobian induces an involution $\sigma$ on $\widetilde{\mathcal{X}_g^{(g-1)}}$. This is given as follows: If $(P_1, \ldots, P_{g-1})$ denotes a section of $\widetilde{\mathcal{X}_g^{(g-1)}} \to \mathcal{M}'_g$, then $\sigma(P_1, \ldots, P_{g-1})$ is the unique section $(R_1, \ldots, R_{g-1})$ of $\widetilde{\mathcal{X}_g^{(g-1)}} \to \mathcal{M}'_g$, such that the sum $P_1 + \cdots + P_{g-1} + R_1 + \cdots + R_{g-1}$ represents the canonical bundle on $\mathcal{X}_g/\mathcal{M}'_g$. Now the integral can be computed as follows

$$\int_{pr_{g+1}} \gamma^*(\delta_{\Theta_\alpha})\gamma^*\omega_0^g = \sum_{j=1}^{g} \int_{pr_{g+1}} \delta_{\{P_j = P_{g+1}\}}\gamma^*\omega_0^g + \int_{pr_{g+1}} \delta_{\{P_g \in \sigma(P_1,\ldots,P_{g-1})\}}\gamma^*\omega_0^g, \tag{4.2.9}$$

where $P_g \in \sigma(P_1, \ldots, P_{g-1})$ means, that $\sigma(P_1, \ldots, P_{g-1}) = (R_1, \ldots, R_{g-1})$ and $P_g = R_j$ for some $j \le g-1$. For the terms in the sum we get by symmetry

$$\int_{pr_{g+1}} \delta_{\{P_j=P_{g+1}\}}\gamma^*\omega_0^g = \int_{pr_{g+1}} \delta_{\{P_g=P_{g+1}\}}\gamma^*\omega_0^g = \int_{pr_g} s_{g+1,g}^*\gamma^*\omega_0^g,$$

where the last integral is with respect to the fibres of $pr_g\colon \mathcal{X}_g^g \to \mathcal{X}_g$. We define the following line bundle on $\mathcal{X}_g^g$

$$\widetilde{\mathcal{L}}' = \bigotimes_{j=1}^{g-1} pr_j^*T \otimes \bigotimes_{j<k}^{g-1} pr_{j,k}^*\mathcal{O}(\Delta)$$

and set $\mathcal{L}' = \widetilde{\mathcal{L}}'^{\otimes 8} \otimes T^{\langle 2\rangle}$. Since it holds $s_{g+1,g}^*(\mathcal{L}^{\langle 2\rangle}) = \mathcal{L}'$, we obtain $s_{g+1,g}^*\gamma^*\omega_0 = s_{g+1,g}^*(-\frac{1}{8}c_1(\mathcal{L}^{\langle 2\rangle})) = -\frac{1}{8}c_1(\mathcal{L}')$.

Thus, we compute

$$\int_{pr_g} s_{g+1,g}^*\gamma^*\omega_0^g = (-1)^g \cdot \left(\int_{pr_g} c_1(\widetilde{\mathcal{L}}')^g + \tfrac{g}{8}\int_{pr_g} c_1(\widetilde{\mathcal{L}}')^{g-1} \wedge e_1^A\right).$$

Since the restriction of $e_1^A$ to a fibre of $\pi_g\colon \mathcal{X}_g^g \to \mathcal{M}_g'$ is trivial and the restriction of $c_1(\widetilde{\mathcal{L}}')$ to a fibre $X^{g-1}$ of $pr_g$ is equal to $-\Phi_{g-1}^*\nu_{\mathrm{Jac}(X)}$, where $\Phi_{g-1}$ is the map

$$\Phi_{g-1}\colon X^{g-1} \to \mathrm{Jac}(X), \quad (P_1,\dots,P_{g-1}) \mapsto P_1 + \cdots + P_{g-1} - \alpha_X,$$

we conclude

$$\int_{pr_g} s_{g+1,g}^*\gamma^*\omega_0^g = (-1)^g \cdot \int_{pr_g} c_1(\widetilde{\mathcal{L}}')^g - \tfrac{g}{8}\cdot (g-1)!\cdot g!\cdot e_1^A.$$

Next, we compute the second term of the right hand side of (4.2.9). For this purpose, we define the map

$$\widetilde{\sigma}\colon \widetilde{\mathcal{X}_g^{(g-1)}} \times_{\mathcal{M}_g'} \mathcal{X}_g^2 \to \widetilde{\mathcal{X}_g^{(g-1)}} \times_{\mathcal{M}_g'} \mathcal{X}_g^2,$$
$$((P_1,\dots,P_{g-1}),P_g,P_{g+1}) \mapsto (\sigma(P_1,\dots,P_{g-1}),P_g,P_{g+1}).$$

If we shortly write $\gamma_\sigma = \gamma' \circ \widetilde{\sigma} \circ (\rho_{g-1} \times \mathrm{id}_{\mathcal{X}_g^2})$, we obtain

$$\int_{pr_{g+1}} \delta_{\{P_g\in\sigma(P_1+\cdots+P_{g-1})\}}\gamma^*\omega_0^g = \sum_{j=1}^{g-1}\int_{pr_{g+1}} \delta_{\{P_j=P_g\}}\gamma_\sigma^*\omega_0^g. \qquad (4.2.10)$$

Since $\gamma_\sigma$ is the map

$$\gamma_\sigma\colon \mathcal{X}_g^{g+1} \to \mathcal{U}_g,$$
$$[X; P_1, \dots, P_{g+1}] \mapsto [\mathrm{Jac}(X); -P_1 - \dots - P_{g-1} + P_g - P_{g+1} + \alpha_X],$$

we again apply [dJo14b, Proposition 6.3] to compute $\gamma_\sigma^*\omega_0 = -\frac{1}{8}c_1(\mathcal{N})$, where $\mathcal{N}$ denotes the line bundle $\mathcal{N} = \widetilde{\mathcal{N}}^{\otimes 8} \otimes T^{\langle 2\rangle}$ with

$$\widetilde{\mathcal{N}} = \bigotimes_{\substack{1\le j\le g+1\\ j\neq g}} pr_j^* T \otimes \bigotimes_{\substack{1\le j<k\le g+1\\ j\neq g, k\neq g}} pr_{j,k}^*\mathcal{O}(\Delta) \otimes \bigotimes_{\substack{1\le j\le g+1\\ j\neq g}} pr_{j,g}\mathcal{O}(\Delta)^\vee.$$

Further, we denote the following line bundle on $\mathcal{X}_g^g$

$$\widetilde{\mathcal{N}'} = \bigotimes_{\substack{1\le j\le g\\ j\neq g-1}} pr_j^* T \otimes \bigotimes_{\substack{1\le j<k\le g\\ j\neq g-1, k\neq g-1}} pr_{j,k}^*\mathcal{O}(\Delta)$$

and set $\mathcal{N}' = \widetilde{\mathcal{N}'}^{\otimes 8} \otimes T^{\langle 2\rangle}$. Let $s$ be the section of $pr^g\colon \mathcal{X}_g^{g+1} \to \mathcal{X}_g^g$ defined by

$$s\colon \mathcal{X}_g^g \to \mathcal{X}_g^{g+1}, \quad [X; P_1, \dots, P_g] \mapsto [X; P_1, \dots, P_{g-2}, P_{g-1}, P_{g-1}, P_g].$$

As for $\mathcal{L}'$, we obtain $(\gamma_\sigma \circ s)^*\omega_0 = -\frac{1}{8}c_1(\mathcal{N}')$. Since $c_1(\mathcal{N}')$ does not depend on the $(g-1)$-th factor of $\mathcal{X}_g^g$, we conclude

$$\int_{pr_{g+1}} \delta_{\{P_{g-1}=P_g\}}\gamma_0^*\omega_0^g = \int_{pr_g} s^*\gamma_\sigma^*\omega^g = -\tfrac{1}{8}\int_{pr_g} c_1(\mathcal{N}')^g = 0.$$

By symmetry the entire sum in (4.2.10) vanishes.

By (4.2.1) we obtain $\int_{pr_{g+1}} c_1(\widetilde{\mathcal{L}})^{g+1} = c_1\left(\widetilde{\mathcal{L}}^{\langle g+1\rangle}\right)$, where the Deligne pairing is with respect to the family $pr_{g+1}\colon \mathcal{X}_g^{g+1} \to \mathcal{X}_g$. Likewise, we have $\int_{pr_g} c_1(\widetilde{\mathcal{L}}')^g = c_1\left(\widetilde{\mathcal{L}}'^{\langle g\rangle}\right)$, where the Deligne pairing is with respect to the family $pr_g\colon \mathcal{X}_g^g \to \mathcal{X}_g$. If we apply all results from this section to the equation (4.1.3), we obtain the following relation

$$\tfrac{1}{\pi i}\partial\overline{\partial}H(X) = \tfrac{1}{2}\omega_{\mathrm{Hdg}} - \tfrac{1}{8}e_1^A + \tfrac{(-1)^{g+1}}{(g!)^2}\left(c_1(\widetilde{\mathcal{L}}^{\langle g+1\rangle}) + g\cdot c_1(\widetilde{\mathcal{L}}'^{\langle g\rangle})\right) \qquad (4.2.11)$$

of forms on $\mathcal{M}_g'$. Thus, we have to calculate the forms $c_1(\widetilde{\mathcal{L}}^{\langle g+1\rangle})$ and $c_1(\widetilde{\mathcal{L}}'^{\langle g\rangle})$.

## 4.3 Graphs and Terms

We compute $c_1(\widetilde{\mathcal{L}}^{\langle g+1\rangle})$ and $c_1(\widetilde{\mathcal{L}'}^{\langle g\rangle})$ by associating a graph to each term in the expansions of the powers $\widetilde{\mathcal{L}}^{\langle g+1\rangle}$ and $\widetilde{\mathcal{L}'}^{\langle g\rangle}$. First, we define for $n \le g$ the sets

$$\mathfrak{L}_n = \{pr_j^*T, pr_{j,g+1}^*\mathcal{O}(\Delta), pr_{k,l}^*\mathcal{O}(\Delta) | 1 \le j \le n, 1 \le k < l \le n\},$$

$$\mathfrak{L}'_n = \{pr_j^*T, pr_{k,l}^*\mathcal{O}(\Delta) | 1 \le j \le n, 1 \le k < l \le n\}.$$

For any $(n+1)$-tuple $(\mathcal{L}_0, \dots, \mathcal{L}_n) \in \mathfrak{L}_n^{n+1}$ we define the associated graph $\Gamma(\mathcal{L}_0, \dots, \mathcal{L}_n)$ as follows: The set of vertices is $\{v_1, \dots, v_n, v_{g+1}\}$ and there are $(n+1)$ edges, for every $0 \le j \le n$ either the loop $e_j = (v_k, v_k)$ if it holds $\mathcal{L}_j = pr_k^*T$ or the edge $e_j = (v_k, v_l)$ if $\mathcal{L}_j = pr_{k,l}^*\mathcal{O}(\Delta)$. Further, we define the graph $\Gamma'(\mathcal{L}_0, \dots, \mathcal{L}_n)$ for any $(n+1)$-tuple $(\mathcal{L}_0, \dots, \mathcal{L}_n) \in \mathfrak{L}_n'^{n+1}$ as the graph $\Gamma(\mathcal{L}_0, \dots, \mathcal{L}_n)$ without the vertex $v_{g+1}$.

**Lemma 4.3.1.** *There are constants $a_1, a_2, a_3, a'_1, a'_2, a'_3 \in \mathbb{Z}$ such that we have the following equalities of forms on $\mathcal{X}_g$:*

*(a)* $c_1\left(\widetilde{\mathcal{L}}^{\langle g+1\rangle}\right) = a_1 \cdot \int_{\pi_2} h^3 + a_2 \cdot e^A + a_3 \cdot e_1^A,$

*(b)* $c_1\left(\widetilde{\mathcal{L}'}^{\langle g\rangle}\right) = a'_1 \cdot \int_{\pi_2} h^3 + a'_2 \cdot e^A + a'_3 \cdot e_1^A.$

*Proof.* We only prove (a). The proof of (b) can be done in a very similar way. By linearity, it is enough to show

$$c_1(\langle \mathcal{L}_0, \dots, \mathcal{L}_g\rangle) \in \mathbb{Z} \cdot \int_{\pi_2} h^3 + \mathbb{Z} \cdot e^A + \mathbb{Z} \cdot e_1^A$$

for all $\mathcal{L}_0, \dots, \mathcal{L}_g \in \mathfrak{L}_g$. Write $\Gamma_1, \dots, \Gamma_r$ for the connected components of $\Gamma(\mathcal{L}_0, \dots \mathcal{L}_g)$, where $\Gamma_1$ is the connected component containing the vertex $v_{g+1}$. Denote by $g_j$ the first Betti number of $\Gamma_j$ for $1 \le j \le r$. We have $\sum_{j=1}^r g_j = r$. If we had $g_j = 0$ for some $j \ge 2$, we would obtain $c_1(\langle \mathcal{L}_0, \dots, \mathcal{L}_g\rangle) = 0$ by (4.2.4). Hence, we distinguish the following two cases:

- In the first case we have $g_j = 1$ for all $1 \le j \le r$. By symmetry we can assume that the edges contained in $\Gamma_1$ are the edges associated to $\mathcal{L}_0, \dots, \mathcal{L}_q$ and that $\mathcal{L}_0, \dots, \mathcal{L}_q \in \mathfrak{L}_q$. If $q < g$, we factorize the family $pr_{g+1}\colon \mathcal{X}_g^{g+1} \to \mathcal{X}_g$ over $pr_{1,\dots,q+1,g+1}\colon \mathcal{X}_g^{g+1} \to \mathcal{X}_g^{q+2}$. Then we obtain by (4.2.2)

$$\langle \mathcal{L}_0, \dots, \mathcal{L}_g\rangle = \langle \mathcal{L}_0, \dots, \mathcal{L}_q, \langle \mathcal{L}_{q+1}, \dots, \mathcal{L}_g\rangle(\mathcal{X}_g^{g+1}/\mathcal{X}_g^{q+2})\rangle(\mathcal{X}_g^{q+2}/\mathcal{X}_g).$$

If we again factorize the family $pr_{q+2}\colon \mathcal{X}_g^{q+2} \to \mathcal{X}_g$ by the projection $pr^{q+1}\colon \mathcal{X}_g^{q+2} \to \mathcal{X}_g^{q+1}$, we can apply (4.2.3) to the right hand side of the equality, such that we get

$$\langle \mathcal{L}_0, \ldots, \mathcal{L}_g\rangle = \deg(\langle \mathcal{L}_{q+1}, \ldots, \mathcal{L}_g\rangle(\mathcal{X}_g^{q+1}/\mathcal{X}_g^{q+2}))\langle \mathcal{L}_0, \ldots, \mathcal{L}_q\rangle(\mathcal{X}_g^{q+1}/\mathcal{X}_g).$$

Hence, we only have to consider $\langle \mathcal{L}_0, \ldots, \mathcal{L}_q\rangle$. The associated graph $\Gamma_1 = \Gamma(\mathcal{L}_0, \ldots, \mathcal{L}_q)$ is connected and has first Betti number $g_1 = 1$. If $v_j$ with $1 \le j \le q$ is a vertex of $\Gamma_1$ with $\deg(v_j) = 1$, then we may assume, that $e_q$ is the unique edge connected to $v_j$ and we obtain by (4.2.3)

$$\langle \mathcal{L}_0, \ldots, \mathcal{L}_q\rangle(\mathcal{X}_g^{q+1}/\mathcal{X}_g) = \deg(\mathcal{L}_q)\cdot\langle \mathcal{L}_0, \ldots, \mathcal{L}_{q-1}\rangle(\mathcal{X}_g^{q}/\mathcal{X}_g),$$

where we factorize the family $pr_{q+1}\colon \mathcal{X}_g^{q+1} \to \mathcal{X}_g$ by the projection $pr^j\colon \mathcal{X}_g^{q+1} \to \mathcal{X}_g^q$. The associated graph $\Gamma(\mathcal{L}_0, \ldots, \mathcal{L}_{q-1})$ is obtained from $\Gamma_1$ by removing the vertex $v_j$ and the edge $e_q$.

If $v_j$ with $1 \le j \le q$ is a vertex of $\Gamma_1$ with $\deg(v_j) = 2$, we may assume, that $e_{q-1}$ and $e_q$ are the edges connected to $v_j$. Now we get by (4.2.2)

$$\langle \mathcal{L}_0, \ldots, \mathcal{L}_q\rangle(\mathcal{X}_g^{q+1}/\mathcal{X}_g) = \langle \mathcal{L}_0, \ldots, \mathcal{L}_{q-2}, \langle \mathcal{L}_{q-1}, \mathcal{L}_q\rangle(\mathcal{X}_g^{q+1}/\mathcal{X}_g^q)\rangle(\mathcal{X}_g^q/\mathcal{X}_g),$$

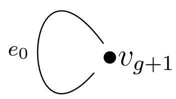
where we again factorize the family $pr_{q+1}\colon \mathcal{X}_g^{q+1} \to \mathcal{X}_g$ by the projection $pr^j\colon \mathcal{X}_g^{q+1} \to \mathcal{X}_g^q$. The line bundles $\mathcal{L}_{q-1}$ and $\mathcal{L}_q$ have to be equal to $pr_{k_1,j}^*\mathcal{O}(\Delta)$, respectively $pr_{k_2,j}^*\mathcal{O}(\Delta)$, for some $k_1, k_2 \in \{1, \ldots, q, g+1\}$. Hence, we have by a similar computation as for (4.2.6) and (4.2.7)

$$\langle \mathcal{L}_{q-1}, \mathcal{L}_q\rangle = \langle pr_{k_1,j}^*\mathcal{O}(\Delta), pr_{k_2,j}^*\mathcal{O}(\Delta)\rangle = pr_{k_1,k_2}^*\mathcal{O}(\Delta)$$

if $k_1 \ne k_2$, and

$$\langle \mathcal{L}_{q-1}, \mathcal{L}_q\rangle = \langle pr_{k_1,j}^*\mathcal{O}(\Delta), pr_{k_1,j}^*\mathcal{O}(\Delta)\rangle = pr_{k_1}^* T$$

if $k_1 = k_2$. Thus, the associated graph $\Gamma(\mathcal{L}_0, \ldots, \mathcal{L}_{q-2}, \langle \mathcal{L}_{q-1}, \mathcal{L}_q\rangle)$ is well defined and it arises from $\Gamma_1$ by removing the vertex $v_j$ and replacing the edges $e_{q-1}$ and $e_q$ by an edge connecting the two not necessarily different neighbours of $v_j$. Therefore, we can assume that the vertices $v_1, \ldots, v_q$ of $\Gamma_1$ have degree at least 3. This is only possible if $\Gamma_1$ only consists of the vertex $v_{g+1}$ and the loop $e_0 = (v_{g+1}, v_{g+1})$:



.

This means, that we always have in this case

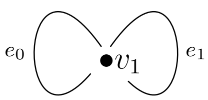
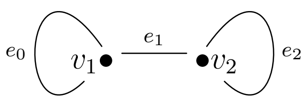
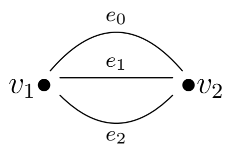
$$c_1(\langle \mathcal{L}_0, \ldots, \mathcal{L}_g \rangle) = n \cdot c_1(T) = n \cdot e^A$$

for some $n \in \mathbb{Z}$.

- The second case can be handled very similarly to the first one. Here, we have $g_1 = 0$, $g_k = 2$ for some $2 \leq k \leq r$ and $g_j = 1$ for $j \notin \{1, k\}$. Again, we may assume by symmetry that the edges contained in $\Gamma_k$ are the edges associated to $\mathcal{L}_0, \ldots, \mathcal{L}_q$ and that $\mathcal{L}_0, \ldots, \mathcal{L}_q \in \mathfrak{L}'_q$. As in the first case we get by (4.2.2) and (4.2.3)

$$\langle \mathcal{L}_0, \ldots, \mathcal{L}_g \rangle = \deg(\langle \mathcal{L}_{q+1}, \ldots, \mathcal{L}_g \rangle) \cdot \langle \mathcal{L}_0, \ldots, \mathcal{L}_q \rangle.$$

By the same arguments as in the first case, we can reduce to the case, where the vertices $v_1, \ldots, v_q$ of $\Gamma_k$ have degree at least 3. But these are all vertices of $\Gamma_k$. Moreover, $\Gamma_k$ is connected and its first Betti number is 2. Hence, there are up to permutations only the possibilities

(a)  (b)  (c) 


for $\Gamma_k$.

The graph (a) corresponds to $q = 1$ and $\mathcal{L}_0 = \mathcal{L}_1 = T$. In this case we have $c_1(\langle \mathcal{L}_0, \mathcal{L}_1 \rangle) = e_1^A$ by (4.2.1). The graph (b) corresponds to $q = 2$, $\mathcal{L}_0 = pr_1^* T$, $\mathcal{L}_1 = pr_{1,2}^* \mathcal{O}(\Delta)$ and $\mathcal{L}_2 = pr_2^* T$. We can apply (4.2.2) to obtain

$$\langle pr_1^* T, pr_{1,2}^* \mathcal{O}(\Delta), pr_2^* T \rangle = \langle \langle pr_1^* T, pr_{1,2}^* \mathcal{O}(\Delta) \rangle, T \rangle,$$

where we factorize the family $\pi_2 \colon \mathcal{X}_g^2 \to \mathcal{M}'_g$ by $pr_2 \colon \mathcal{X}_g^2 \to \mathcal{X}_g$. Since

$$\langle pr_1^* T, pr_{1,2}^* \mathcal{O}(\Delta) \rangle = T,$$

we again conclude $c_1(\langle \mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2 \rangle) = e_1^A$. Finally, the graph (c) corresponds to $q = 2$ and $\mathcal{L}_0 = \mathcal{L}_1 = \mathcal{L}_2 = pr_{1,2}^* \mathcal{O}(\Delta)$. Hence, we can again apply (4.2.1) to obtain $c_1(\langle \mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2 \rangle) = \int_{\pi_2} h^3$.

□

By the lemma and by formula (4.2.11) we have the following equality of forms on $\mathcal{M}'_g$

$$\tfrac{1}{\pi i}\partial\overline{\partial}H(X) = \tfrac{1}{2}\omega_{\mathrm{Hdg}} + b_1 \cdot \int_{\pi_2} h^3 + b_2 \cdot e_1^A + b_3 \cdot e^A, \tag{4.3.1}$$

where $X$ is non-hyperelliptic and $b_1, b_2, b_3 \in \mathbb{Q}$ are constants depending only on $g$. Since $H(X)$ does not depend on the choice of the point $P_{g+1}$, we must have $b_3 = 0$. Since the constants only depend on $g$ and we assume $g \geq 3$, the formula is also true for hyperelliptic Riemann surfaces by continuity. If we restrict (4.3.1) to the hyperelliptic locus $\mathcal{H}_g[2]$, we know by Theorem 3.3.1 and by formulas (4.1.1) and (4.1.2) that

$$b_1 \cdot \int_{\pi_2} h^3 + b_2 \cdot e_1^A = \tfrac{1}{12}\int_{\pi_2} h^3 - \tfrac{1}{8}e_1^A.$$

But on $\mathcal{H}_g[2]$ we have the linear dependence $3e_1^A = (2-2g)\int_{\pi_2} h^3$, see [dJo14b, Proposition 10.7]. Hence, we can only conclude $b_2 = \frac{12b_1 - g}{8(g-1)}$. Therefore, we have to compute $b_1$ in another way. This is done by the following lemma.

**Lemma 4.3.2.** *The constants $a_1$ and $a'_1$ in Lemma 4.3.1 satisfy $a_1 = 0$ and $a'_1 = \frac{g!(g-1)!}{12}(-1)^{g-1}$.*

*Proof.* We have to weight and count the graphs associated to the terms in the expansions of the powers $c_1(\widetilde{\mathcal{L}}^{\langle g+1\rangle})$ and $c_1(\widetilde{\mathcal{L}'}^{\langle g\rangle})$. We first consider the case $c_1(\widetilde{\mathcal{L}}^{\langle g+1\rangle})$. By the arguments of the proof of Lemma 4.3.1 we have $c_1(\langle\mathcal{L}_0,\dots,\mathcal{L}_g\rangle) = a \cdot \int_{\pi_2} h^3$ for some $\mathcal{L}_0,\dots,\mathcal{L}_g \in \mathfrak{L}'_g$ and $a \neq 0$ only if the associated graph $\Gamma(\mathcal{L}_0,\dots,\mathcal{L}_g)$ has a subgraph $\Gamma_0$ containing two different vertices $v_j, v_k$ which are connected by three disjunct paths not involving the vertex $v_{g+1}$.

Hence, we can compute $a_1$ by

$$a_1 = \sum_{k=3}^{g+1} A_k \cdot B_{g,g+1-k} \cdot \binom{g}{k-1} \cdot \binom{g+1}{k}, \tag{4.3.2}$$

where $A_k$ is the number of $k$-tuples $(\mathcal{L}_0,\dots,\mathcal{L}_{k-1}) \in \mathfrak{L}'^k_{k-1}$ such that the graph $\Gamma'(\mathcal{L}_0,\dots,\mathcal{L}_{k-1})$ has two vertices of degree 3 and all other vertices have degree 2, that means it is of the form $\Gamma_0$ described above. To define $B_{g,k}$ we introduce another graph $\Gamma_g(\mathcal{L}_0,\dots,\mathcal{L}_q)$ for any $(q+1)$-tuple $(\mathcal{L}_0,\dots,\mathcal{L}_q) \in \mathfrak{L}_g^{q+1}$, which is defined as follows: The set of vertices is $\{v_1,\dots,v_{g+1}\}$ and there are $q+1$ edges associated to $\mathcal{L}_0,\dots,\mathcal{L}_q$ in the same way as for $\Gamma$. Now $B_{g,k}$ is the sum of the weights $w(\mathcal{L}_0,\dots,\mathcal{L}_{k-1})$ associated to all $k$-tuples $(\mathcal{L}_0,\dots,\mathcal{L}_{k-1}) \in \mathfrak{L}_g^k$, where the weight is defined as follows: $w(\mathcal{L}_0, \ldots, \mathcal{L}_{k-1})$ is 0 if for some $l \leq k$ and some subset $\{j_1, \ldots, j_l\} \subseteq \{0, \ldots, k-1\}$ of cardinality $l$ at most $l-1$ of the vertices $v_1, \ldots, v_k$ of the graph $\Gamma_g(\mathcal{L}_{j_1}, \ldots, \mathcal{L}_{j_l})$ has non-zero degree and otherwise it has the value $(2-2g)^{b_1} \cdot (-1)^{\deg(v_{g+1})}$, where $b_1$ denotes the first betti number of $\Gamma_g(\mathcal{L}_0, \ldots, \mathcal{L}_{k-1})$. Note, that if we define another weight $w'$ in exactly the same way except that we replace the vertices $v_1, \ldots, v_k$ by $v_{g-k+1}, \ldots, v_g$, we will obtain the same number $B_{g,k}$ by symmetry. In particular, if the graph $\Gamma(\mathcal{L}_0, \ldots, \mathcal{L}_{k-1})$ is of the form $\Gamma_0$ and if we have $w'(\mathcal{L}_k, \ldots, \mathcal{L}_g) = 0$, then it follows $c_1(\langle \mathcal{L}_0, \ldots, \mathcal{L}_g \rangle) = 0$ by (4.2.4). But for simpler notations we will work with the weight $w$.

We obtain the binomial coefficient $\binom{g}{k-1}$ by choosing $k-1$ of the $g$ vertices $\{v_1, \ldots, v_g\}$ of the associated graph $\Gamma(\mathcal{L}_0, \ldots, \mathcal{L}_g)$ to be the vertices of $\Gamma_0$ and the binomial coefficient $\binom{g+1}{k}$ by choosing the position of the $k$-tuple associated to the graph $\Gamma_0$ in the whole $(g+1)$-tuple $(\mathcal{L}_0, \ldots, \mathcal{L}_g)$.

One can check the correctness of formula (4.3.2) by the methods of proof of Lemma 4.3.1: Every circle in the associated graph of a tuple $(\mathcal{L}_0, \ldots, \mathcal{L}_g)$ outside of $\Gamma_0$ can be reduced to a loop, which is associated to a line bundle $pr_j^*T$ having degree $\deg(pr_j^*T) = 2-2g$. Moreover, every line bundle of the form $pr_{j,g+1}^*\mathcal{O}(\Delta)$ occurs as its dual in $\widetilde{\mathcal{L}}$. Thus, we have to multiply with $\deg(pr_{j,g+1}^*\mathcal{O}(\Delta)^\vee) = -1$ for every line bundle of this form in the tuple $(\mathcal{L}_0, \ldots, \mathcal{L}_g)$. This justifies the formula of the weight $w$ and hence, formula (4.3.2) follows by elementary combinatorics and using (4.2.3) inductively.

**Claim.** *It holds* $A_k = \binom{k-1}{2} \cdot \frac{k!(k-1)!}{12}$ *for* $3 \leq k \leq g+1$.

*Proof of the claim.* Let $k = 3$. Since $(pr_{1,2}^*\mathcal{O}(\Delta), pr_{1,2}^*\mathcal{O}(\Delta), pr_{1,2}^*\mathcal{O}(\Delta))$ is the only tuple with the desired property, we have $A_3 = 1$. Hence, we can assume $k \geq 4$. Write $(\mathcal{L}_0, \ldots, \mathcal{L}_{k-1})$ for a tuple of the desired form and $v_{j_1}, v_{j_2}$ for the vertices of the associated graph, which have degree 3. There are $\binom{k-1}{2}$ possible choices for $v_{j_1}$ and $v_{j_2}$. Further, there are $(k-3)!\binom{k-1}{2}$ choices to order the remaining vertices and to divide them in 3 groups representing the 3 paths from $v_{j_1}$ to $v_{j_2}$. But here, the 3 paths are ordered, so we have to divide by the possibilities to order these paths. We have to distinguish two cases:

- The lengths of two paths from $v_{j_1}$ to $v_{j_2}$ are 1. Then there are 3 possibilities to order all three paths. But on the other side, we have $\frac{k!}{2}$ possibilities to order the line bundles in the tuple $(\mathcal{L}_0, \ldots, \mathcal{L}_{k-1})$, since two of them are equal. Hence, in this case we have to multiply by $\frac{k!}{6}$.

- Otherwise, there are two paths from $v_{j_1}$ to $v_{j_2}$ with length at least 2. Therefore, there are $3! = 6$ possibilities to order all three paths.

Here, we have $k!$ possibilities to order the line bundles in the tuple $(\mathcal{L}_0, \ldots, \mathcal{L}_{k-1})$, since all of them are different. Hence, in this case we also have to multiply by $\frac{k!}{6}$.

Thus, we conclude $A_k = \binom{k-1}{2} \cdot (k-3)! \cdot \binom{k-1}{2} \cdot \frac{k!}{6} = \binom{k-1}{2} \cdot \frac{k!(k-1)!}{12}$. □

**Claim.** *The number $B_{g,k}$ is given by $B_{g,k} = (-1)^k \cdot k! \cdot \frac{g!}{(g-k)!}$.*

*Proof of the claim.* We prove this by induction over $k$. If $k = 0$, we only have the empty tuple, which is weighted by 1. Hence, we assume $k > 0$ and that the claim is true for $k-1$. For any $q \leq g$ denote by $\mathbb{Z}[\mathfrak{L}_g^q]$ the free abelian group over the set of $q$-tuples of elements in $\mathfrak{L}_g$. Write $w_q \in \mathbb{Z}[\mathfrak{L}_g^q]$ for the distinguished element

$$w_q = \sum_{(\mathcal{L}_0,\ldots,\mathcal{L}_{q-1}) \in \mathbb{Z}[\mathfrak{L}_g^q]} w(\mathcal{L}_0, \ldots, \mathcal{L}_{q-1}) \cdot (\mathcal{L}_0, \ldots, \mathcal{L}_{q-1}).$$

For any element $c \in \mathbb{Z}[\mathfrak{L}_g^q]$ we define the degree $\deg(c) \in \mathbb{Z}$ to be the sum of its coefficients. Then we have $B_{g,q} = \deg(w_q)$, and the induction hypothesis states

$$\deg(w_{k-1}) = (-1)^{k-1} \cdot (k-1)! \cdot \tfrac{g!}{(g-k+1)!}.$$

We have to prove $\deg(w_k) = -k(g-k+1) \cdot \deg(w_{k-1})$. We distinguish the following cases to extend a non-zero weighted $(k-1)$-tuple to a non-zero weighted $k$-tuple:

$(1)_k$ $(\mathcal{L}_0, \ldots, \mathcal{L}_{k-2}) \to (\mathcal{L}_0, \ldots, \mathcal{L}_{k-2}, pr_k^* T)$,

$(2)_k$ $(\mathcal{L}_0, \ldots, \mathcal{L}_{k-2}) \to (\mathcal{L}_0, \ldots, \mathcal{L}_{k-2}, pr_{k,l}^* \mathcal{O}(\Delta))$ for any $1 \leq l \leq g$ with $l \neq k$ and $pr_{k,l}^* \mathcal{O}(\Delta) \notin \{\mathcal{L}_0, \ldots, \mathcal{L}_{k-2}\}$,

$(3)_k$ $(\mathcal{L}_0, \ldots, \mathcal{L}_{k-2}) \to (\mathcal{L}_0, \ldots, \mathcal{L}_{k-2}, pr_{k,l}^* \mathcal{O}(\Delta))$ for any $1 \leq l \leq g$ with $l \neq k$ and $pr_{k,l}^* \mathcal{O}(\Delta) \in \{\mathcal{L}_0, \ldots, \mathcal{L}_{k-2}\}$,

$(4)_k$ $(\mathcal{L}_0, \ldots, \mathcal{L}_{i-1}, pr_l^* T, \mathcal{L}_{i+1}, \ldots, \mathcal{L}_{k-2})$
$\to (\mathcal{L}_0, \ldots, \mathcal{L}_{i-1}, pr_{l,k}^* \mathcal{O}(\Delta), \mathcal{L}_{i+1}, \ldots, \mathcal{L}_{k-2}, pr_{k,l}^* \mathcal{O}(\Delta))$ for any $l \neq k$,

$(5)_k$ $(\mathcal{L}_0, \ldots, \mathcal{L}_{i-1}, pr_{l,m}^* \mathcal{O}(\Delta), \mathcal{L}_{i+1}, \ldots, \mathcal{L}_{k-2})$
$\to (\mathcal{L}_0, \ldots, \mathcal{L}_{i-1}, pr_{l,k}^* \mathcal{O}(\Delta), \mathcal{L}_{i+1}, \ldots, \mathcal{L}_{k-2}, pr_{k,m}^* \mathcal{O}(\Delta))$ for any $l \neq k$ and $m \neq k$ with $l \neq m$,

$(6)_k$ $(\mathcal{L}_0, \ldots, \mathcal{L}_{k-2}) \to (\mathcal{L}_0, \ldots, \mathcal{L}_{k-2}, pr_{k,g+1}^* \mathcal{O}(\Delta))$.

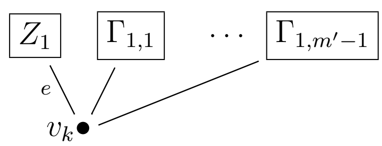
We additionally consider the extensions $(1)_j$-$(6)_j$ for any $1 \leq j \leq k$ which coincide with $(1)_k$-$(6)_k$ with the change that the new line bundle occurs in the $j$-th factor instead of the last factor. In this way, we obtain all $k$-tuples of non-zero weight as extensions of $(k-1)$-tuples of non-zero weight. However, the same $k$-tuple can be constructed by different extensions. Hence, we have to count them with suitable multiplicities.

For a $(k-1)$-tuple $(\mathcal{L}_0, \ldots, \mathcal{L}_{k-2}) \in \mathfrak{L}_g^{k-1}$ we denote by $m = \deg(v_k)$ the degree of the vertex $v_k$ of the associated graph $\Gamma_g(\mathcal{L}_0, \ldots, \mathcal{L}_{k-2})$. Let $w'_k$ be the element in $\mathbb{Z}[\mathcal{L}_g^k]$, which we obtain by taking for all $(k-1)$-tuples $(\mathcal{L}_0, \ldots, \mathcal{L}_{k-2}) \in \mathfrak{L}_g^{k-1}$ and all $j \leq k$

- the extensions $(1)_j$ times $(2-2g) \cdot w(\mathcal{L}_0, \ldots, \mathcal{L}_{k-2})$,
- the extensions $(2)_j$ times $(1-m) \cdot w(\mathcal{L}_0, \ldots, \mathcal{L}_{k-2})$,
- the extensions $(3)_j$ times $(g-m) \cdot w(\mathcal{L}_0, \ldots, \mathcal{L}_{k-2})$,
- the extensions $(4)_j$ and $(5)_j$ times $w(\mathcal{L}_0, \ldots, \mathcal{L}_{k-2})$ and
- the extensions $(6)_j$ times $(m-1) \cdot w(\mathcal{L}_0, \ldots, \mathcal{L}_{k-2})$.

Next, we prove $w'_k = w_k$. Let $(\mathcal{L}_0, \ldots, \mathcal{L}_{k-1}) \in \mathfrak{L}_g^k$ be a $k$-tuple with non-zero weight. Denote by $m' = \deg(v_k)$ the degree of the vertex $v_k$ in the associated graph $\Gamma = \Gamma(\mathcal{L}_0, \ldots, \mathcal{L}_{k-1})$. If $\Gamma$ has a loop at $v_k$, the tuple $(\mathcal{L}_0, \ldots, \mathcal{L}_{k-1})$ can only be obtained by an extension of kind $(1)_j$ from a non-zero weighted $(k-1)$-tuple. Hence, we can assume, that $\Gamma$ has no loop at $v_k$. Since every extension $(1)_j$-$(6)_j$ only adds edges connected to $v_k$, it is enough to consider the connected component $\Gamma_1$ of the graph $\Gamma$, which contains the vertex $v_k$. Its first Betti number $b_1(\Gamma_1)$ is either 1 and all its vertices form a subset of $\{v_1, \ldots, v_k\}$ or $b_1(\Gamma_1) = 0$ and $\Gamma_1$ additionally contains one vertex $v_i$ with $k < i \leq g+1$. More precisely, we distinguish the following four cases, where we denote by $Z_1$ a connected subgraph of $\Gamma_1$ with first Betti number $b_1(Z_1) = 1$ and by $\Gamma_{1,1}, \ldots, \Gamma_{1,m'}$ connected subgraphs of $\Gamma_1$, which are trees. The sets of vertices of $Z_1, \Gamma_{1,1}, \ldots, \Gamma_{1,m'-1}$ are assumed to be non-empty subsets of $\{v_1, \ldots, v_{k-1}\}$ and the set of vertices of $\Gamma_{1,m'}$ is assumed to be a subset of $\{v_1, \ldots, v_{k-1}, v_i\}$, which has to contain $v_i$.

- In the first case, we consider $\Gamma_1$ with $b_1(\Gamma_1) = 1$ and $\Gamma_1$ has the structure



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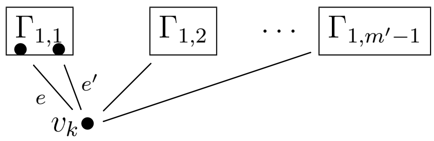
These graphs can be obtained by an extension of the form $(2)_j$, where the edge $e$ is added, or by an extension of the form $(5)_j$, where an edge from $Z_1$ to $\Gamma_{1,l}$ for some $l \leq m' - 1$ is replaced by the edges from $Z_1$ to $v_k$ and from $\Gamma_{1,l}$ to $v_k$. In both cases the weight of the tuple is preserved by the extension and we have $m = m' - 1$ for the extension of the form $(2)_j$. Here, the $j$ is unique by the choice of the $k$-tuple and the kind of extension. All in all, the coefficient of the $k$-tuple $(\mathcal{L}_0, \ldots, \mathcal{L}_{k-1})$ in $w'_k$ equals

$$((1 - (m' - 1)) + (m' - 1)) \cdot w(\mathcal{L}_0, \ldots, \mathcal{L}_{k-1}) = w(\mathcal{L}_0, \ldots, \mathcal{L}_{k-1}).$$

- Next, we consider graphs $\Gamma_1$ with $b_1(\Gamma_1) = 1$ and having the structure

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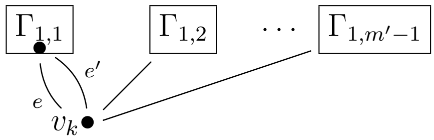
Here, there are two edges from $v_k$ to the subgraph $\Gamma_{1,1}$ landing in two different vertices. These graphs can be obtained by extensions of the form $(2)_j$, where the edge $e$ or the edge $e'$ is added, or by an extension of the form $(5)_j$, where an edge from one of the neighbours of $v_k$ in the graph $\Gamma_1$ to another is replaced by two edges connecting each of these two neighbours with $v_k$, where at least one of the neighbours has to be contained in $\Gamma_{1,1}$. Hence, there are 2 possible extensions of the form $(2)_j$, where $m = m' - 1$, and $(m' - 1) + (m' - 2)$ possible extensions of the form $(5)_j$. The weight of the tuple is preserved by these extensions. The $j$ is unique by the choice of the $k$-tuple and the kind of extension. Hence, the coefficient of the $k$-tuple $(\mathcal{L}_0, \ldots, \mathcal{L}_{k-1})$ in $w'_k$ is given by

$$(2(1-(m'-1))+(m'-1+m'-2))w(\mathcal{L}_0, \ldots, \mathcal{L}_{k-1}) = w(\mathcal{L}_0, \ldots, \mathcal{L}_{k-1}).$$

- We have a third case with $b_1(\Gamma_1) = 1$, where $\Gamma_1$ is of the form

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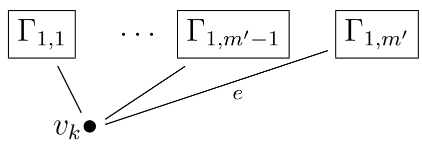
Here, there are two edges from $v_k$ to the subgraph $\Gamma_{1,1}$ landing in the same vertex. These graphs can be obtained by an extension of the form $(3)_j$, where the edge $e$ or the edge $e'$ is added, by an extension of the form $(4)_j$, where a loop is replaced by the edges $e$ and $e'$, or by an extension of the form $(5)_j$, where an edge from one of the neighbours of $v_k$ in the graph $\Gamma_1$ to another is replaced by two edges connecting each of the two neighbours with $v_k$, where one of the neighbours has to be the one in $\Gamma_{1,1}$. The extensions of the form $(3)_j$ and $(5)_j$ multiply the weight by $(2-2g)$, the extensions of the form $(4)_j$ preserve the weight and we have $m = m'-1$ for the extensions of the form $(3)_j$. Since each of these extensions adds at least one of the edges $e$ and $e'$, which represent two isomorphic line bundles in the tuple $(\mathcal{L}_0, \ldots, \mathcal{L}_{k-1})$, there are two choices for the $j$. Therefore, the coefficient of the $k$-tuple $(\mathcal{L}_0, \ldots, \mathcal{L}_{k-1})$ in $w'_k$ is

$$2\left(\tfrac{g-(m'-1)}{2-2g} + 1 + \tfrac{m'-2}{2-2g}\right) \cdot w(\mathcal{L}_0, \ldots, \mathcal{L}_{k-1}) = w(\mathcal{L}_0, \ldots, \mathcal{L}_{k-1}).$$

- Finally, it remains the case $b_1(\Gamma_1) = 0$ and $\Gamma_1$ is of the form




.

There is an $1 \le r \le g+1$ with $r \ne k$ such that $e = (v_k, v_r)$. We additionally distinguish the following two cases.

(a) Assume $r \le g$. Then we obtain graphs of this form by an extension of the form $(2)_j$, where the edge $e$ is added, or by an extension of the form $(5)_j$, where an edge from $v_r$ to $\Gamma_{1,l}$ for some $l < m'$ is replaced by the edge from $\Gamma_{1,l}$ to $v_k$ and the edge $(v_k, v_r)$. These extensions preserve the weights of the corresponding tuples. Further, we have $m = m'-1$ for the extension $(2)_j$. The $j$ is unique by the choice of the $k$-tuple and the kind of extension. Hence, the coefficient of the $k$-tuple $(\mathcal{L}_0, \ldots, \mathcal{L}_{k-1})$ in $w'_k$ equals

$$((1-(m'-1)) + (m'-1)) \cdot w(\mathcal{L}_0, \ldots, \mathcal{L}_{k-1}) = w(\mathcal{L}_0, \ldots, \mathcal{L}_{k-1}).$$

(b) Otherwise, we have $r = i = g+1$. Then graphs of this form can be obtained by extensions of the form $(5)_j$, where an edge from $v_r$ to $\Gamma_{1,l}$ for some $l < m'$ is replaced by the edge from $\Gamma_{1,l}$ to $v_k$ and the edge $(v_k, v_r)$, or by an extension of the form $(6)_j$, where the edge $e$ is added. The extensions of the form $(5)_j$ preserve the weight, while the extension of the form $(6)_j$ changes the weight by the factor $-1$. Further, we have $m = m'-1$ for the extension of the form $(6)_j$. The $j$ is unique by the choice of the $k$-tuple and the kind of extension. Thus, the coefficient of the $k$-tuple $(\mathcal{L}_0, \ldots, \mathcal{L}_{k-1})$ in $w'_k$ is given by

$$((m'-1)+(-1)\cdot((m'-1)-1))\cdot w(\mathcal{L}_0, \ldots, \mathcal{L}_{k-1}) = w(\mathcal{L}_0, \ldots, \mathcal{L}_{k-1}).$$

Thus, we obtain $w'_k = w_k$. We conclude that $\deg(w_k) = \deg(w_{k-1}) \cdot c(g,k)$, where $c(g,k)$ equals

$$\begin{aligned} & k \cdot ((2-2g) + (1-m)(g-1-m) + (g-m)m + (k-1-m) + (m-1)) \\ = & -k(g-k+1). \end{aligned}$$

This proves the claim. □

Now we can prove the first equation of the lemma by putting the values for $A_k$ and $B_{g,k}$ into equation (4.3.2)

$$\begin{aligned} a_1 &= \sum_{k=3}^{g+1} \tbinom{k-1}{2} \tfrac{k!(k-1)!}{12} (-1)^{g+1-k} (g+1-k)! \tfrac{g!}{(k-1)!} \tbinom{g}{k-1} \tbinom{g+1}{k} \\ &= \tfrac{g!(g+1)!}{12} (-1)^g \sum_{k=2}^{g} (-1)^k \tbinom{k}{2} \tbinom{g}{k} = 0. \end{aligned}$$

For the last equality see for example [BQ08].

For $a'_1$ we obtain by the same arguments

$$a'_1 = \sum_{k=3}^{g} A_k \cdot B'_{g,g-k} \cdot \tbinom{g-1}{k-1} \cdot \tbinom{g}{k}.$$

Here, $B'_{g,k}$ denotes the sum of the weights $w(\mathcal{L}_0, \ldots, \mathcal{L}_{k-1})$ for all $k$-tuples $(\mathcal{L}_0, \ldots, \mathcal{L}_{k-1}) \in \mathfrak{L}'^{k}_{g-1}$. We obtain $B'_{g,k} = (-1)^k \cdot k! \cdot \frac{g!}{(g-k)!}$ in the same way as for $B_{g,k}$. One only has to note, that there is no extension of the form $(6)_j$ and we have $l, m \leq g-1$ for all extensions $(2)_j - (5)_j$. To calculate $a'_1$, we claim the following identity of binomial coefficients

$$\sum_{k=3}^{n} (-1)^{k-1} \tbinom{k-1}{2} \tbinom{n}{k} = 1, \qquad (4.3.3)$$

where $n \geq 3$. We prove this by induction over $n$. It is trivially true for $n = 3$. Hence, we can assume $n \geq 4$. If (4.3.3) is true for $n-1$, we obtain

$$\begin{aligned} & \sum_{k=3}^{n} (-1)^{k-1} \tbinom{k-1}{2} \tbinom{n}{k} \\ = & \sum_{k=3}^{n} (-1)^{k-1} \tbinom{k-1}{2} \tbinom{n-1}{k-1} + \sum_{k=3}^{n-1} (-1)^{k-1} \tbinom{k-1}{2} \tbinom{n-1}{k} = 0 + 1. \end{aligned}$$

For the vanishing of the first sum see again [BQ08]. The second sum is 1 by the induction hypothesis. Now we get for $a_1'$

$$\begin{aligned} a_1' &= \sum_{k=3}^{g} \tbinom{k-1}{2} \tfrac{k!(k-1)!}{12} (-1)^{g-k} (g-k)! \tfrac{g!}{k!} \tbinom{g-1}{k-1} \tbinom{g}{k} \\ &= \tfrac{g!(g-1)!}{12} (-1)^{g-1} \sum_{k=3}^{g} (-1)^{k-1} \tbinom{k-1}{2} \tbinom{g}{k} = \tfrac{g!(g-1)!}{12} (-1)^{g-1}. \end{aligned}$$

This completes the proof of the lemma. □

Now we can compute the constants in (4.3.1). Equation (4.2.11) and Lemma 4.3.2 yield $b_1 = \frac{1}{12}$ and hence, $b_2 = -\frac{1}{8}$. We summarize this to

$$\tfrac{1}{\pi i} \partial\overline{\partial} H(X) = \tfrac{1}{2}\omega_{\mathrm{Hdg}} + \tfrac{1}{12} \int_{\pi_2} h^3 - \tfrac{1}{8} e_1^A \tag{4.3.4}$$

as forms on $\mathcal{M}_g[2]$. Since all these forms are already defined on $\mathcal{M}_g$, this formula also holds for the corresponding forms on $\mathcal{M}_g$.

## 4.4 Main result

In this section we deduce our main result, which generalizes the second formula in Theorem 3.3.1 to compact and connected Riemann surfaces. Precisely, we prove the following theorem.

**Theorem 4.4.1.** *Any compact and connected Riemann surface $X$ of genus $g \geq 1$ satisfies $\delta(X) = -24H(X) + 2\varphi(X) - 8g \log 2\pi$.*

*Proof.* For $g = 1$ and $g = 2$ this follows from [Fal84, Section 7], respectively Theorem 3.3.1. Thus, we assume $g \geq 3$. Consider the function

$$f(X) = \delta(X) + 24H(X) - 2\varphi(X)$$

as a real-valued function on $\mathcal{M}_g$. By (4.1.1), (4.1.2) and (4.3.4) this function satisfies $\partial\overline{\partial} f(X) = 0$, that means $f$ is harmonic on $\mathcal{M}_g$. But there are no non-constant holomorphic functions on $\mathcal{M}_g$, see for example [ACG11, p. 437]. Hence, there are also no non-constant harmonic functions on $\mathcal{M}_g$. Thus, $f(X)$ is constant on $\mathcal{M}_g$, and we obtain $f(X) = -8g \log 2\pi$ by Theorem 3.3.1. □

As an application of the theorem we obtain a lower bound for the invariant $\delta(X)$ by applying the lower bounds in Proposition 1.1.1 and (1.2.9).

**Corollary 4.4.2.** *For any compact and connected Riemann surface $X$ of genus $g \geq 1$ we have $\delta(X) > -2g \log 2\pi^4$.*

One can generalize the invariant $\|\Delta_g\|$ of hyperelliptic Riemann surfaces to arbitrary compact and connected Riemann surfaces of positive genus, even to principally polarised complex abelian varieties. Let $(A, \Theta)$ be any principally polarised complex abelian variety of dimension $g \geq 1$ as in Section 1.1. We define the set $\mathcal{D}_2 = \{z \in A \setminus \Theta \mid 2z = 0\}$ and we set

$$\|\Delta_g\|(A, \Theta) = 2^{-4(g+1)\binom{2g}{g-1}} \sum_{\substack{\mathcal{J} \subseteq \mathcal{D}_2 \\ \#\mathcal{J} = r}} \prod_{z \in \mathcal{J}} \|\theta\|(z)^8,$$

where $r = \binom{2g+1}{g+1}$. In particular, we have $\|\Delta_g\|(\mathrm{Jac}(X)) = \|\Delta_g\|(X)$ if $X$ is a hyperelliptic Riemann surface of genus $g \geq 2$. Hence, we also define $\|\Delta_g\|(X) = \|\Delta_g\|(\mathrm{Jac}(X))$ if $X$ is an arbitrary connected and compact Riemann surface of genus $g \geq 1$. However, the first formula of Theorem 3.3.1 and Corollaries 3.3.2 and 3.3.3 are not true for arbitrary connected and compact Riemann surfaces. Indeed, we have $\frac{1}{\pi i}\partial\overline{\partial} \log \|\Delta_g\|(X) = 4r \cdot \omega_{\mathrm{Hdg}} - \delta_Z$ as forms on $\mathcal{M}_g$, where $Z \subseteq \mathcal{M}_g$ is the vanishing locus of $\|\Delta_g\|$, which is known to be empty at least for $g \leq 5$, see [Bea13, Section 5]. Comparing this with the forms (4.1.1), (4.1.2) and (4.3.4), we notice that each of the mentioned formulas implies $3e_1^A = (2 - 2g) \int_{\pi_2} h^3$, which is not true in general on $\mathcal{M}_g$, see also [dJo14b, Section 10].

## 4.5 Bounds for theta functions

In this section we give an upper bound for the function $\|\theta\|$. This bound will be used in the next section to obtain an upper bound for the Arakelov–Green function.

**Lemma 4.5.1.** *Let $(A, \Theta)$ be any principally polarised complex abelian variety of dimension $g \geq 1$ as in Section 1.1. For any real number $r > 0$ and any $z \in A$ we obtain*

$$\log \|\theta\|(z) + rH(A, \Theta) \leq \tfrac{1}{4} \left(5 \max(2, g^3)(1 + r) + g^2 \left(r + 2 + \tfrac{1}{r}\right)\right) \log 2.$$

*Proof.* The idea of the proof is based on [Gra00, Section 2.3.2]. We denote by $\mathbb{H}_g$ the Siegel upper half-space. The symplectic group $\mathrm{Sp}(2g, \mathbb{R})$ acts on $\mathbb{C}^g \times \mathbb{H}_g$ by

$$(z, \Omega) \mapsto ({}^{\mathrm{t}}(C\Omega + D)^{-1} z, (A\Omega + B)(C\Omega + D)^{-1})$$

for any $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2g, \mathbb{R})$. The group $\mathrm{Sp}(2g, \mathbb{Z})$ acts by translation by 2-torsion points on $\|\theta\|$, see for example [BL04, Theorem 8.6.1]. Hence, $\sup_{z \in A} \|\theta\|(z)$ is independent of a representative $\Omega$ in a coset of $\mathrm{Sp}(2g, \mathbb{Z})$. Therefore, it suffices to prove the assertion for a fundamental domain in $\mathbb{H}_g$. We define the fundamental domain $\mathcal{F}_g$ to be the subspace of matrices $\Omega \in \mathbb{H}_g$ satisfying the following bounds:

(i) For all $1 \leq j, k \leq g$, we have $|(\mathrm{Re}\ \Omega)_{jk}| \leq \frac{1}{2}$.

(ii) For all $\gamma \in \mathrm{Sp}(2g, \mathbb{Z})$, we have $\det(\mathrm{Im}\ (\gamma \cdot \Omega)) \leq \det(\mathrm{Im}\ \Omega)$.

(iii) For all $n \in \mathbb{Z}^g$ and $j \leq g$, such that $n_j, \ldots, n_g$ are relatively prime, we have the inequalities ${}^{\mathrm{t}}n(\mathrm{Im}\ \Omega)n \geq (\mathrm{Im}\ \Omega)_{jj}$.

(iv) For all $j \leq g - 1$, we have $(\mathrm{Im}\ \Omega)_{j,j+1} \geq 0$.

It follows for example from [Igu72, Chapter V.§4.] that this is indeed a fundamental domain in $\mathbb{H}_g$. Let $z \in \mathbb{C}^g$. If we write $y = \mathrm{Im}\ z = (\mathrm{Im}\ \Omega) \cdot b$ for some $b \in \mathbb{R}^g$, then the triangle inequality gives

$$\exp(-\pi {}^{\mathrm{t}}y(\mathrm{Im}\ \Omega)^{-1}y) \cdot |\theta(z, \Omega)| \leq \sum_{n \in \mathbb{Z}^g} \exp(-\pi {}^{\mathrm{t}}(n + b)(\mathrm{Im}\ \Omega)(n + b)).$$

Let $c(g) = \left(\frac{4}{g^3}\right)^{g-1} \left(\frac{3}{4}\right)^{g(g-1)/2}$ be the Minkowski constants. By (iii) and (iv), $\Omega$ is Minkowski reduced and we obtain the Minkowski inequality

$${}^{\mathrm{t}}m(\mathrm{Im}\ \Omega)m \geq c(g) \sum_{j=1}^{g} m_j^2 (\mathrm{Im}\ \Omega)_{jj}$$

for all $m \in \mathbb{R}^g$. Hence, we obtain

$$\sum_{n \in \mathbb{Z}^g} \exp(-\pi {}^{\mathrm{t}}(n + b)(\mathrm{Im}\ \Omega)(n + b)) \leq \prod_{j=1}^{g} \sum_{n \in \mathbb{Z}} \exp(-\pi c(g)(\mathrm{Im}\ \Omega)_{jj}(n + b_j)^2).$$

Since we sum over $\mathbb{Z}$, we can assume $0 \leq b_j \leq 1$ for all $1 \leq j \leq g$. We have

$$(n + b_j)^2 \geq n + b_j^2 \geq n \qquad \text{for } n \geq 0,$$
$$(n + b_j)^2 \geq -n - 1 + (1 - b_j)^2 \geq -n - 1 \qquad \text{for } n \leq -1.$$

This allows us to split the sum into two sums over $\mathbb{N}_0$ and bound in the following way:

$$\sum_{n \in \mathbb{Z}} \exp(-\pi c(g)(\mathrm{Im}\ \Omega)_{jj}(n + b_j)^2) \leq 2 \sum_{n \in \mathbb{N}_0} \exp(-\pi c(g)(\mathrm{Im}\ \Omega)_{jj} \cdot n)$$
$$\leq \frac{2}{1 - \exp(-\pi c(g)(\mathrm{Im}\ \Omega)_{jj})}.$$

We have $(\mathrm{Im}\ \Omega)_{jj} \geq \sqrt{3}/2$ for all $j \leq g$, see also [Igu72, Chapter V.§4.]. Using $e^x \geq x+1$ for $x \in \mathbb{R}$, we get for the function $\theta$:

$$\exp(-\pi{}^{\mathrm{t}}y(\mathrm{Im}\ \Omega)^{-1}y) \cdot |\theta(z,\Omega)| \leq \left(2 + \frac{4}{\pi c(g)\sqrt{3}}\right)^g \leq \max(4, 2^{g^3}). \quad (4.5.1)$$

Here, we used $c(g) \geq 2^{-g^2}$.

Now we consider the case $\frac{1}{4} \leq b_g \leq \frac{3}{4}$. By Hadamard's theorem we have $\det(\mathrm{Im}\ \Omega) \leq \prod_{j=1}^g (\mathrm{Im}\ \Omega)_{jj}$ and hence, $\det(\mathrm{Im}\ \Omega) \leq ((\mathrm{Im}\ \Omega)_{gg})^g$ by property (iii). Therefore, we can bound for $s \geq 0$

$$\begin{aligned}
&\det(\mathrm{Im}\ \Omega)^s \sum_{n\in\mathbb{Z}} \exp(-\pi c(g)(\mathrm{Im}\ \Omega)_{gg}(n+b_g)^2)\\
\leq& 2((\mathrm{Im}\ \Omega)_{gg})^{gs} \sum_{n\in\mathbb{N}_0} \exp\left(-\pi c(g)(\mathrm{Im}\ \Omega)_{gg} \cdot \left(n + \tfrac{1}{16}\right)\right)\\
\leq& ((\mathrm{Im}\ \Omega)_{gg})^{gs} \exp\left(-\tfrac{1}{16}\pi c(g)(\mathrm{Im}\ \Omega)_{gg}\right)\left(2 + \frac{4}{\pi c(g)\sqrt{3}}\right).
\end{aligned}$$

Using again $e^x \geq x+1$, we get

$$((\mathrm{Im}\ \Omega)_{gg})^{gs} \exp\left(-\tfrac{1}{16}\pi c(g)(\mathrm{Im}\ \Omega)_{gg}\right) \leq \left(\frac{16gs}{e\pi c(g)}\right)^{gs} \leq 2^{g^3 s + 2g^2 s^2}.$$

Putting these bounds together, we get for the case $\frac{1}{4} \leq b_g \leq \frac{3}{4}$:

$$\left(s - \tfrac{1}{4}\right)\log\det(\mathrm{Im}\ \Omega) + \log\|\theta\|(z) \leq (\max(2, g^3) + g^3 s + 2g^2 s^2) \cdot \log 2.$$

To integrate over $A$ means to integrate over a fundamental domain in $\mathbb{C}^g$ for the lattice $\mathbb{Z}^g + \Omega\mathbb{Z}^g$. Hence, we can again assume $0 \leq b_g \leq 1$ and by the translation invariance of the volume form $\nu^g$ we get

$$\frac{1}{g!}\int_{z\in A|b_g\in[1/4,3/4]} \nu^g(z) = \frac{1}{g!}\int_{z\in A|b_g\in[0,1/4]\cup[3/4,1]} \nu^g(z) = \frac{1}{2}$$

Therefore, we split and bound the integral $H(A,\Theta)$ in the following way:

$$\begin{aligned}
&\left(s - \tfrac{1}{4}\right)\log\det(\mathrm{Im}\ \Omega) + H(A,\Theta)\\
\leq& \tfrac{1}{2}\max(2, g^3)\log 2\\
&+ \frac{1}{g!}\int_{z\in A|1/4\leq b_g\leq 3/4} \left(\left(2s - \tfrac{1}{4}\right)\log\det(\mathrm{Im}\ \Omega) + \log\|\theta\|(z)\right)\nu^g(z)\\
\leq& (\max(2, g^3) + g^3 s + 4g^2 s^2)\log 2.
\end{aligned}$$

Now the lemma follows by combining this with equation (4.5.1) and setting $s = (r+1)/(4r)$. □

## 4.6 The Arakelov–Green function

We give an explicit expression for the Arakelov–Green function by calculating Bost's invariant $A(X)$ in (1.2.2). Furthermore, we will bound the supremum of the Arakelov–Green function in terms of $\delta(X)$ and we give another expression for $\delta$. Let $X$ be any compact and connected Riemann surface of genus $g \geq 2$.

**Theorem 4.6.1.** *It holds*

$$g(P,Q) = \tfrac{1}{g!}\int_{\Theta+P-Q} \log\|\theta\|\nu^{g-1} + \tfrac{1}{2g}\varphi(X) - H(X).$$

*Proof.* Integrating (1.2.2) with $\mu(P)$ gives

$$-A(X) = \tfrac{1}{g!}\int_X \left(\int_{\Theta+P-Q} \log\|\theta\|\nu^{g-1}\right)\mu(P).$$

We define the map

$$\Phi_\Theta \colon X^{g-1} \to \Theta, \quad (P_1,\dots,P_{g-1}) \mapsto P_1 + \cdots + P_{g-1},$$

which is smooth, surjective and generically of degree $(g-1)!$. Since $\nu$ is translation-invariant, we conclude that

$$-A(X) = \tfrac{1}{(g-1)!g!}\int_{X^g} \log\|\theta\|(P_1+\cdots+P_g-Q)\Phi_\Theta^*\nu^{g-1}(P_1,\dots,P_{g-1})\mu(P_g).$$

For a divisor $D \in \Theta^{sm}$ and points $P_g, Q \in X$ the term

$$\log\|\Lambda\|(D) = \log\|\theta\|(D+P_g-Q) - g(P_g,Q) - g(D,Q) - g(\sigma(D),P_g) \tag{4.6.1}$$

does not depend on $P_g$ or $Q$, see [dJo08, Proposition 4.3]. We obtain

$$\begin{aligned}-A(X) =& \tfrac{1}{(g-1)!g!}\int_{X^{g-1}} \log\|\Lambda\|(P_1+\cdots+P_{g-1})\Phi_\Theta^*\nu^{g-1}(P_1,\dots,P_{g-1})\\ =& \tfrac{1}{(g!)^2}\int_{X^g} \log\|\Lambda\|(P_1+\cdots+P_{g-1})\Phi^*\nu^g(P_1,\dots,P_g),\end{aligned}$$

since the Arakelov–Green functions in (4.6.1) integrates to 0. The latter equality follows by Lemma 2.2.1. If we again substitute $\log\|\Lambda\|$ by (4.6.1) in the last expression, only the integral of $\log\|\theta\|(P_1+\cdots+P_g-Q)$ and the integral of $-g(\sigma(P_1+\cdots+P_{g-1}),P_g)$ are non-zero. The first one gives $H(X)$ and the second one equals $-\frac{1}{2g}\varphi(X)$ by Lemma 2.2.3. Thus, we obtain the identity $A(X) = \frac{1}{2g}\varphi(X) - H(X)$. □

As a corollary we bound the Arakelov–Green function in terms of $\delta(X)$.

**Corollary 4.6.2.** *The Arakelov–Green function is bounded by $\delta(X)$ in the following way:*

$$\sup_{P,Q\in X} g(P,Q) < \begin{cases} \frac{1}{4g}\delta(X) + 3g^3\log 2 & \text{if } g \leq 5, \\ \frac{2g+1}{48g}\delta(X) + 2g^3\log 2 & \text{if } g > 5. \end{cases}$$

*Proof.* Since $\int_{\Theta+P-Q}\nu^{g-1} = g!$, Lemma 4.5.1 with $r = 1/(2g)$ yields

$$\frac{1}{g!}\int_{\Theta+P-Q}\log\|\theta\|\nu^{g-1} + \frac{1}{2g}H(X) \leq \left(\tfrac{7}{4}g^3 + \tfrac{9}{8}g^2 + \tfrac{1}{8}g\right)\log 2.$$

For $g \leq 5$ we have by Theorem 4.4.1 and Proposition 1.1.1

$$\frac{1}{2g}\varphi(X) - H(X) - \frac{1}{2g}H(X) < \frac{1}{4g}\delta(X) - \frac{11-2g}{8}\log 2 + 2\log 2\pi,$$

while we obtain for $g > 5$ using the bound (1.2.9)

$$\frac{1}{2g}\varphi(X) - H(X) - \frac{1}{2g}H(X) < \frac{2g+1}{48g}\delta(X) + \frac{2g+1}{6}\log 2\pi.$$

If we apply these inequalities to the expression for the Arakelov–Green function in Theorem 4.6.1, we get the estimates in the corollary. □

Next, we discuss an application of this bound. Let $\mathcal{L}$ be an admissible line bundle on $X$, that means $\mathcal{L}$ is equipped with a hermitian metric and it holds $\partial\overline{\partial}\log\|s\|^2 = 2\pi i\deg(\mathcal{L})\mu$ for a local generating section $s \in H^0(X,\mathcal{L})$. Faltings introduced in [Fal84, Section 3] a canonical metric on the determinant of cohomology

$$\lambda(\mathbf{R}\Gamma(X,\mathcal{L})) = \bigwedge^{\max} H^0(X,\mathcal{L}) \otimes \bigwedge^{\max} H^1(X,\mathcal{L})^{\otimes -1}$$

for all admissible line bundles $\mathcal{L}$, which is given up to a common scalar factor. We choose this factor, such that we have on $\lambda(\mathbf{R}\Gamma(X,\Omega_X^1)) = \bigwedge^g H^0(X,\Omega_X^1)$ the metric induced by (1.2.3).

If $\deg\mathcal{L} = r+g-1$ with $r \geq g$, the metric on $\lambda(\mathbf{R}\Gamma(X,\mathcal{L}))$ gives a volume form on $H^0(X,\mathcal{L})$. Let $E$ be a divisor with $\mathcal{O}_X(E) \cong \mathcal{L}$. There are points $P_1,\dots,P_r$ on $X$, such that $\mathcal{O}_X(E-(P_1+\cdots+P_r))$ has no global sections. The canonical norm on $\lambda(\mathbf{R}\Gamma(X,\mathcal{O}_X(E-(P_1+\cdots+P_r)))) \cong \mathbb{C}$ is given by the real number $\|\theta\|(E-(P_1+\cdots+P_r))^{-1}\cdot\exp(-\delta(X)/8)$, see [Fal84, p. 402].

Faltings proved that for every $\epsilon > 0$ there exists a constant $d(\epsilon)$ such that for any line bundle $\mathcal{L}$ of degree $d \geq d(\epsilon)$ the volume of the unit ball under the $L^2$-norm can be estimated in the following way

$$V(\mathcal{L}) = \mathrm{Vol}\left(\left\{f \in H^0(X,\mathcal{L}) \mid \int_X \|f\|^2\mu \leq 1\right\}\right) \geq \exp(-\epsilon d^2),$$

see [Fal84, Theorem 2]. In the proof he used an upper bound for the Arakelov–Green function. With our bound in Corollary 4.6.2 we obtain the following more explicit, but asymptotically worse result.

**Corollary 4.6.3.** *Any admissible line bundle $\mathcal{L}$ on $X$ of degree $r + g - 1$ with $r \geq g$ satisfies*

$$\log V(\mathcal{L}) \geq \begin{cases} -\frac{1}{48g}\delta(X) - r^2\left(\frac{1}{4g}\delta(X) + 3g^3\log 2\right) & \text{if } g \leq 5, \\ -r^2\left(\frac{2g+1}{48g}\delta(X) + 2g^3\log 2\right) & \text{if } g > 5. \end{cases}$$

*Proof.* It follows from the proof of [Fal84, Theorem 2] that

$$\frac{1}{V(\mathcal{L})} = \pi^{-r}\int_{X^r} v(P_1,\dots,P_r)^{-1}\prod_{j\neq k} G(P_j,P_k)\mu(P_1)\dots\mu(P_r),$$

where $v(P_1,\dots,P_r) = \|\theta\|(E-(P_1+\dots+P_r))^{-2}\cdot\exp(-\delta(X)/4)$ is the volume form on $\lambda(\mathbf{R}\Gamma(X,\mathcal{O}_X(E-(P_1+\dots+P_r)))) \cong \mathbb{C}$ with the notation as above. Hence, we can bound

$$\log V(\mathcal{L}) \geq r\log\pi - \tfrac{1}{4}\delta(X) - 2\log\sup_{z\in\mathrm{Jac}(X)}\|\theta\|(z) - r(r-1)\cdot\sup_{P,Q\in X} g(P,Q).$$

Applying the bounds in Lemma 4.5.1 with $r = 1/(4g)$ to $\sup_{z\in\mathrm{Jac}(X)}\|\theta\|(z)$ and the bound in Corollary 4.6.2 to $\sup_{P,Q\in X} g(P,Q)$ and using that we have the inequality $H(X) \geq -\frac{1}{24}\delta(X) - \frac{g}{3}\log 2\pi$ by Theorem 4.4.1, we obtain the estimate in the corollary. □

As an application of the proof of Theorem 4.6.1, we obtain a formula for $\delta(X)$ only in terms of integrals of the function $\log\|\theta\|$.

**Corollary 4.6.4.** *We have*

$$\delta(X) = -\tfrac{4g}{g!}\int_X\left(\int_{\Theta+P-Q}\log\|\theta\|\nu^{g-1}\right)\mu(P) + (4g-24)H(X) - 8g\log 2\pi.$$

*Proof.* By the proof of Theorem 4.6.1 we have

$$-\tfrac{4g}{g!}\int_X\left(\int_{\Theta+P-Q}\log\|\theta\|\nu^{g-1}\right)\mu(P) = 2\varphi(X) - 4gH(X).$$

If we apply this to Theorem 4.4.1, we obtain the corollary. □

# Chapter 5

# The case of abelian varieties

We state formulas for $\delta(X)$ and $\varphi(X)$ only in terms of $H(X)$ and $\Lambda(X)$, such that we obtain canonical extensions of the functions $\delta$ and $\varphi$ to the moduli space of indecomposable principally polarised complex abelian varieties. Further, we discuss some of the asymptotics of these extensions.

## 5.1 The delta invariant of abelian varieties

We deduce the following expressions for $\delta$ and $\varphi$ from the expressions in Theorem 4.4.1 and formula (1.2.6).

**Theorem 5.1.1.** *For any compact and connected Riemann surface $X$ of genus $g \geq 2$, the invariant $\delta(X)$ satisfies*

$$\delta(X) = 2(g-7)H(X) - 2\Lambda(X) - 4g\log 2\pi.$$

*Further, the invariant $\varphi(X)$ satisfies*

$$\varphi(X) = (g+5)H(X) - \Lambda(X) + 2g\log 2\pi.$$

*Proof.* If we integrate the logarithm of formula (1.2.6) with respect to $\Phi^*\nu^g$, we obtain by equation (2.1.2) and by Lemma 2.2.2

$$\tfrac{1}{(g!)^2}\int_{X^g} \log\|\eta\|(P_1+\cdots+P_{g-1})\Phi^*\nu^g = (g-1)H(X) - \tfrac{1}{4}\delta(X) - \tfrac{1}{2}\varphi(X).$$

Denote by $\Phi_\Theta$ the map defined in Section 4.6. We have

$$\begin{aligned}\Lambda(X) &= \tfrac{1}{(g-1)!g!}\int_{X^{g-1}} \log\|\eta\|(P_1+\cdots+P_{g-1})\Phi_\Theta^*\nu^{g-1}\\ &= \tfrac{1}{(g!)^2}\int_{X^g} \log\|\eta\|(P_1+\cdots+P_{g-1})\Phi^*\nu^g,\end{aligned}$$

where the latter equality follows from Lemma 2.2.1. Putting both equations together, we obtain

$$\Lambda(X) = (g-1)H(X) - \tfrac{1}{4}\delta(X) - \tfrac{1}{2}\varphi(X).$$

Now both formulas in the theorem follow by Theorem 4.4.1. □

Let $(A, \Theta)$ be an indecomposable principally polarised complex abelian variety of dimension $g \geq 2$ as in Section 1.1. We define

$$\begin{aligned} \delta(A,\Theta) &= 2(g-7)H(A,\Theta) - 2\Lambda(A,\Theta) - 4g\log 2\pi, \\ \varphi(A,\Theta) &= (g+5)H(A,\Theta) - \Lambda(A,\Theta) + 2g\log 2\pi. \end{aligned}$$

Then we have $\delta(\mathrm{Jac}(X)) = \delta(X)$ and $\varphi(\mathrm{Jac}(X)) = \varphi(X)$ for any compact and connected Riemann surface $X$ by Theorem 5.1.1. Hence, we obtain canonical extensions of $\delta$ and $\varphi$ to the moduli space of indecomposable principally polarised complex abelian varieties. For Riemann surfaces we have the bounds $\varphi(X) > 0$ and $\delta(X) > -2g\log 2\pi^4$. It is a natural question whether these bounds are still true for the extended versions of $\delta$ and $\varphi$.

**Question 5.1.2.** *Do all indecomposable principally polarised complex abelian varieties $(A, \Theta)$ of dimension $g \geq 2$ satisfy $\varphi(A,\Theta) > 0$?*

If the answer of this question is yes, we will also obtain the lower bound $\delta(A,\Theta) > -2g\log 2\pi^4$. If the answer is no, $\varphi$ could be seen as an indicator for an abelian variety to be a Jacobian.

Finally in this section, we consider the Hain–Reed invariant $\beta_g(X)$ of any compact and connected Riemann surface $X$ of genus $g \geq 2$, which we already mentioned in the introduction. This invariant is only defined modulo constants on $\mathcal{M}_g$. De Jong obtained a canonical normalization by proving that a representative of $\beta_g(X)$ is given by $\frac{1}{3}((2g-2)\varphi(X) + (2g+1)\delta(X))$, see [dJo13, Theorem 1.4]. Hence, we can also define $\beta_g$ for indecomposable principally polarised complex abelian varieties by

$$\beta_g(A,\Theta) = 2(g-4)(g+1)H(A,\Theta) - 2g\Lambda(A,\Theta) - \tfrac{4g(g+2)}{3}\log 2\pi.$$

By Theorem 5.1.1 we have $\beta_g(\mathrm{Jac}(X)) = \beta_g(X)$ for any compact and connected Riemann surface $X$ of genus $g \geq 2$.

## 5.2 Asymptotics

Next, we discuss some of the asymptotics of the extended versions of the invariants $\delta$ and $\varphi$ for degenerating families of indecomposable principally polarised complex abelian varieties. We denote by $D \subseteq \mathbb{C}$ the open unit disc. Further, we denote $f(t) = O(g(t))$ for two functions $f, g\colon D \to \mathbb{R}$ if there exists a bound $M \in \mathbb{R}$ not depending on $t$, such that $|f(t)| \leq M \cdot |g(t)|$ for all $t \in D$. If $\mathscr{X} \to D$ is a family of complex curves, such that $\mathscr{X}_t$ is a Riemann surface if and only if $t \neq 0$ and $\mathscr{X}_0$ has exactly one node, then Jorgenson [Jor90], Wentworth [Wen91] and de Jong [dJo14a] showed that $\delta(\mathscr{X}_t)$ and $\varphi(\mathscr{X}_t)$ go to infinity for $t \to 0$. By continuity, $\delta$ and $\varphi$ have to be infinity on the boundary of $\mathcal{M}_g$ in its Deligne–Mumford compactification $\overline{\mathcal{M}}_g$ and hence, they are bounded from below on $\mathcal{M}_g$.

It is a natural question, whether the same is true for the extended versions of $\delta$ and $\varphi$ on the moduli space of indecomposable principally polarised complex abelian varieties. As a first step, we obtain the following asymptotic behaviour of $\delta$ and $\varphi$ for families of indecomposable principally polarised complex abelian varieties degenerating to a decomposable principally polarised complex abelian variety.

**Proposition 5.2.1.** *Let $\tau\colon D \to \mathbb{H}_g$ be a holomorphic embedding and write $(A_t, \Theta_t)$ for the principally polarised complex abelian variety associated to $\tau(t)$. If $(A_t, \Theta_t)$ is indecomposable for $t \neq 0$ and $(A_0, \Theta_0)$ is the product of two indecomposable principally polarised complex abelian varieties $(A_1, \Theta_1)$ and $(A_2, \Theta_2)$ of positive dimensions $g_1$, respectively $g_2$, then it holds*

$$\lim_{t\to 0} H(A_t, \Theta_t) = H(A_1, \Theta_1) + H(A_2, \Theta_2),$$

$$\Lambda(A_t, \Theta_t) - \tfrac{2g_1g_2}{g} \log|t| = O(1),$$
$$\delta(A_t, \Theta_t) + \tfrac{4g_1g_2}{g} \log|t| = O(1) \text{ and}$$
$$\varphi(A_t, \Theta_t) + \tfrac{2g_1g_2}{g} \log|t| = O(1).$$

*Proof.* For $t \in D$ and $j \in \{1, 2\}$ we denote by $\nu_t = \nu_{(A_t,\Theta_t)}$ and $\nu_j = \nu_{(A_j,\Theta_j)}$ the canonical $(1, 1)$ form of $(A_t, \Theta_t)$ respectively $(A_j, \Theta_j)$. We may assume, that $\tau(0)$ is of the form

$$\tau(0) = \begin{pmatrix} \Omega_1 & 0 \\ 0 & \Omega_2 \end{pmatrix},$$

where $\Omega_j \in \mathbb{H}_{g_j}$ is a matrix associated to $(A_j, \Theta_j)$. We have $\nu_0 = \nu_1 + \nu_2$ and hence,

$$\tfrac{1}{g!}\nu_0^g = \tfrac{1}{g_1!g_2!}\nu_1^{g_1}\nu_2^{g_2} \quad \text{and} \quad \tfrac{1}{g!}\nu_0^{g-1} = \tfrac{g_1}{g\cdot g_1!g_2!}\nu_1^{g_1-1}\nu_2^{g_2} + \tfrac{g_2}{g\cdot g_1!g_2!}\nu_1^{g_1}\nu_2^{g_2-1}.$$

Likewise, we obtain $\det(\mathrm{Im}\,\tau(0)) = \det(\mathrm{Im}\,\Omega_1) \cdot \det(\mathrm{Im}\,\Omega_2)$. Every $z \in A_t$ can be represented by $a + \tau(t) \cdot b$ for some real vectors $a, b \in [-\frac{1}{2}, \frac{1}{2}]^g$. Fix arbitrary vectors $a, b \in [-\frac{1}{2}, \frac{1}{2}]^g$ and write $z_t = a + \tau(t) \cdot b$. We obtain for the function $\theta$

$$\begin{aligned} &\exp\left(-\pi {}^{\mathrm{t}}(\mathrm{Im}\ z_t)(\mathrm{Im}\ \tau(t))^{-1}(\mathrm{Im}\ z_t)\right) \cdot |\theta|(\tau(t); z_t) \\ &= \left| \sum_{n \in \mathbb{Z}^g} \exp\left(\pi i {}^{\mathrm{t}}(n+b)\tau(t)(n+b) + 2\pi i {}^{\mathrm{t}}n a\right) \right|. \end{aligned}$$

In particular, we have $\|\theta\|(\tau(0); \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}) = \|\theta\|(\Omega_1; z_1) \cdot \|\theta\|(\Omega_2; z_2)$, where $z_j \in \mathbb{C}^{g_j}$, and hence, $H(A_0, \Theta_0) = H(A_1, \Theta_1) + H(A_2, \Theta_2)$.

We also deduce, that $\Theta_0 = (\Theta_1 \times A_2) \cup (A_1 \times \Theta_2)$. Set for easier notation $n_{g+1} = \frac{1}{2\pi i}$. The function $\|\eta\|$ can be written by

$$\begin{aligned} &\|\eta\|(\tau(t); z_t) \cdot \det(\mathrm{Im}\ \tau(t))^{-(g+5)/4} \\ &= \left| \det\left(4\pi^2 \sum_{n \in \mathbb{Z}^g} n_j n_k \exp(\pi i {}^{\mathrm{t}}(n+b)\tau(t)(n+b) + 2\pi i {}^{\mathrm{t}}n a)\right)_{j,k \leq g+1} \right|, \end{aligned} \tag{5.2.1}$$

where $z_t = a + \tau(t) \cdot b \in \Theta_t$. Write $a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$, where $a_j$ and $b_j$ are $g_j$-dimensional vectors. Let $a + \tau(0) \cdot b$ represent an element in $\Theta_1 \times A_2$. Consider the expression

$$\widetilde{\theta}_{jk}(\tau(t); a, b) = \sum_{n \in \mathbb{Z}^g} n_j n_k \exp(\pi i {}^{\mathrm{t}}(n+b)\tau(t)(n+b) + 2\pi i {}^{\mathrm{t}}n a).$$

If $j \leq g_1$ or $k \leq g_1$, then $\widetilde{\theta}_{jk}(\tau(0); a, b)$ is non-zero for a dense subset of pairs $(a, b)$ in

$$M = \left\{(a, b) \in \left[-\tfrac{1}{2}, \tfrac{1}{2}\right]^g \mid a + \tau(0) \cdot b \in \Theta_1 \times A_2\right\}.$$

Otherwise it is zero, since we can write it as a product containing the factor

$$\sum_{n \in \mathbb{Z}^{g_1}} \exp(\pi i {}^{\mathrm{t}}(n+b_1)\Omega_1(n+b_1) + 2\pi i {}^{\mathrm{t}}n a_1), \tag{5.2.2}$$

which vanishes by $(a_1 + \Omega_1 \cdot b_1) \in \Theta_1$. But the expression

$$\lim_{t \to 0} \frac{\widetilde{\theta}_{jk}(\tau(t); a, b)}{t},$$

is non-zero for a dense subset of pairs $(a, b)$ in $M$. To check this, one uses the chain rule to obtain a linear combination of partial derivations of (5.2.2) with coefficients $\frac{\partial \tau_{pq}(t)}{\partial t}|_{t=0}$ with $p \leq g_1$ and $q > g_2$, which do not vanish all by the definition of $\tau$.

We have to compute the order of vanishing at $t = 0$ for the summands in the expansion of the determinant in (5.2.1). Let $\sigma \in \mathrm{Sym}(g+1)$ be any permutation with $\sigma(g+1) \neq g+1$. Denote by $m(\sigma)$ the cardinality of $\{j \leq g_1 \mid \sigma(j) > g_1\}$. The observations above shows, that

$$\prod_{j=1}^{g+1} \widetilde{\theta}_{j,\sigma(j)}(\tau(t); a, b)$$

vanishes of order $g_2 + 1 - m(\sigma)$ at $t = 0$ for a dense subset of pairs $(a, b)$ in $M$. But for different $j_1, j_2 \leq g_1$ and different $k_1, k_2 > g_1$ the function $\widetilde{\theta}_{j_l k_m}(\tau(t); a, b)$ splits into a product of two factors, such that the expression

$$\widetilde{\theta}_{j_1 k_1}(\tau(t); a, b) \cdot \widetilde{\theta}_{j_2 k_2}(\tau(t); a, b) - \widetilde{\theta}_{j_1 k_2}(\tau(t); a, b) \cdot \widetilde{\theta}_{j_2 k_1}(\tau(t); a, b)$$

vanishes at $t = 0$ of order at least 1. If $\sigma$ satisfies $m(\sigma) \geq 2$, $\sigma(j_1) = k_1$ and $\sigma(j_2) = k_2$, then we construct $\sigma' \in \mathrm{Sym}(g+1)$ by setting $\sigma'(j_1) = \sigma(j_2)$, $\sigma'(j_2) = \sigma(j_1)$ and $\sigma'(j) = \sigma(j)$ for $j \notin \{j_1, j_2\}$. We obtain that

$$\prod_{j=1}^{g+1} \widetilde{\theta}_{j,\sigma(j)}(\tau(t); a, b) - \prod_{j=1}^{g+1} \widetilde{\theta}_{j,\sigma'(j)}(\tau(t); a, b)$$

vanishes of order at least $g_2 + 2 - m(\sigma)$. Inductively, we deduce that the determinant in (5.2.1) vanishes of order at least $g_2$. Since there is no such cancellation for permutations with $m(\sigma) = 1$, we conclude that

$$\log \|\eta\|(\tau(t); a + \tau(t) \cdot b) = g_2 \log |t| + O(1)$$

for a dense subset of pairs $(a, b)$ in $M$. We can argue analogously for $a$, $b$ with $(a + \tau(0) \cdot b) \in A_1 \times \Theta_2$. Then we obtain for the invariant $\Lambda(A_t, \Theta_t)$:

$$\begin{aligned}
\Lambda(A_t, \Theta_t) = & \int_{\Theta_1 \times A_2} (g_2 \log |t| + O(1)) \tfrac{g_1}{g \cdot g_1! g_2!} \nu_1^{g_1 - 1} \nu_2^{g_2} \\
& + \int_{A_1 \times \Theta_2} (g_1 \log |t| + O(1)) \tfrac{g_2}{g \cdot g_1! g_2!} \nu_1^{g_1} \nu_2^{g_2 - 1} \\
= & \tfrac{2 g_1 g_2}{g} \log |t| + O(1).
\end{aligned}$$

Now the formulas for $\delta$ and $\varphi$ in the proposition follow by Theorem 5.1.1. $\square$

# Chapter 6

# Applications

Finally, we apply our results to Arakelov theory. For details on Arakelov theory we refer to the introduction, and we also continue the notation from the introduction.

## 6.1 Bounds of heights and intersection numbers

We establish some bounds of certain Arakelov intersection numbers and of the heights of points. Let $C \to \mathrm{Spec}\, K$ be a smooth, projective and geometrically connected curve of genus $g \geq 2$ defined over a number field $K$. After a finite field extension, we can assume that $C$ has semi-stable reduction over $B = \mathrm{Spec}\, \mathcal{O}_K$, see [DM69]. Let $p\colon \mathscr{C} \to B$ be the minimal regular model of $C$ over $B$. We set $d = [K : \mathbb{Q}]$ and we write $e(C) = \frac{1}{d}(\omega_{\mathscr{C}/B}, \omega_{\mathscr{C}/B})$ for the stable Arakelov self-intersection number of the relative dualizing sheaf $\omega_{\mathscr{C}/B}$. It does not depend on the choice of $K$. Further, we define the stable Faltings height by $h_F(C) = \frac{1}{d}\widehat{\deg} \det p_*\omega_{\mathscr{C}/B}$ and we shortly write

$$\delta(C) = \tfrac{1}{d} \sum_{\sigma\colon K \to \mathbb{C}} \delta(C_\sigma) \quad \text{and} \quad \Delta(C) = \tfrac{1}{d} \sum_{v \in |B|} \delta_v \log N_v,$$

where the first sum runs over all embeddings $\sigma\colon K \to \mathbb{C}$. Now the arithmetic Noether formula, see formula (1) in the introduction, has the form

$$12 h_F(C) = e(C) + \Delta(C) + \delta(C) - 4g \log 2\pi. \tag{6.1.1}$$

As a direct consequence of the arithmetic Noether formula and Corollary 4.4.2 we get the following inequality.

**Corollary 6.1.1.** *The Arakelov self-intersection number $e(C)$ is bounded by*

$$e(C) < 12h_F(C) + 6g \log 2\pi^2.$$

Further, we also define the stable version $\varphi(C) = \frac{1}{d}\sum_{\sigma\colon K\to\mathbb{C}} \varphi(C_\sigma)$ of the Kawazumi–Zhang invariant and the invariant $H(C) = \frac{1}{d}\sum_{\sigma\colon K\to\mathbb{C}} H(C_\sigma)$.

Next, we consider heights of points. Let $P \in C(\overline{K})$ be any geometric point of $C$, where $\overline{K}$ denotes an algebraic closure of $K$. After a finite field extension, we can assume, that $P$ is already defined over $K$ with $K$ as above. Then we define the stable Arakelov height of $P$ by

$$h(P) = \tfrac{1}{d}(\omega_{\mathscr{C}/B}, \mathcal{O}_{\mathscr{C}}(P)).$$

On the other hand, we write $h_{NT}$ for the Neron-Tate height on $\mathrm{Pic}^0(C)$ and we set $h_{NT}(P) = h_{NT}((2g-2)P - K_C)$, where $K_C$ is the canonical bundle on $C$. It holds

$$h_{NT}(P) \leq 2g(g-1)h(P), \tag{6.1.2}$$

see for example [JvK14, Lemma 4.4]. Denote by $W$ the divisor of Weierstraß points in $C$. After a finite field extension, we may assume, that all Weierstraß points are defined over $K$.

**Proposition 6.1.2.** *The heights of the Weierstraß points on $C$ are bounded by*

$$\max_{P\in W} h(P) \leq \sum_{P\in W} h(P) < (6g^2 + 4g + 2)h_F(C) + 12g^4 \cdot \log 2.$$

*In the summation over $W$ the Weierstraß points are counted with their multiplicity in $W$.*

*Proof.* The first inequality is trivial since it holds $h(P) \geq 0$ for all geometric points $P$ of $C$, see [Fal84, Theorem 5]. It follows from the proof of [dJo09, Theorem 4.3] that the sum $\sum_{P\in W} h(P)$ is bounded by

$$(3g-1)(2g+1)h_F(C) + \tfrac{g+1}{4}e(C) + g(2g-1)(g+1)\log(2\pi) - 2g^2 \log T(C),$$

where the invariant $T(C)$ is defined by $T(C) = \frac{1}{d}\sum_{\sigma\colon K\to\mathbb{C}} T(C_\sigma)$ and

$$\log T(X) = \tfrac{1}{4}\delta(X) - \tfrac{g-1}{g^2} S_1(X)$$

for any compact and connected Riemann surface $X$ of genus $g \geq 2$, see [dJo05a], where one has to pay attention to the following misprint in [dJo05a, Theorem 4.4]: the $g^3$ occurring in the exponent should be $g^2$.

We will bound the invariant $-\log T(X)$. An application of Lemma 4.5.1 with $r = 1/(2g)$ yields

$$\tfrac{g-1}{g^2} S_1(X) + \tfrac{g-1}{2g^3} H(X) < \left(\tfrac{7}{4}g^2 - \tfrac{5}{8}g - 1\right) \log 2.$$

Now we get by Theorem 4.4.1 and the bounds in (1.2.9) and Proposition 1.1.1

$$-\log T(X) < 2g \log 2\pi + \left(\tfrac{7}{4}g^2 - \tfrac{17}{8}g - \tfrac{7}{8}\right) \log 2.$$

If we put this into the bound for $\sum_{P\in W} h(P)$ and if we bound $e(C)$ in terms of $h_F(C)$ by Corollary 6.1.1, we get the inequality in the proposition. $\square$

We apply this bound and the bound of the Arakelov–Green function in Corollary 4.6.2 to obtain the following bound for certain Arakelov intersection numbers.

**Proposition 6.1.3.** *Let $W_1, \ldots, W_g$ be arbitrary and not necessary different Weierstraß points on $C$ and write $D$ for the effective divisor $\sum_{j=1}^{g} W_j$. Further, let $\mathcal{L}$ be any line bundle on $C$ of degree $0$, that is represented by a torsion point in $\mathrm{Pic}^0(C)$ and that satisfies $\dim H^0(\mathcal{L}(D)) = 1$. Write $D'$ for the unique effective divisor on $C$, such that $\mathcal{L} \cong \mathcal{O}_C(D' - D)$. Let $P \in C(\overline{K})$ be any geometric point of $C$. We may assume that $P, D, D'$ and $\mathcal{L}$ are defined over $K$. It holds*

$$\tfrac{1}{d}(D' - D, P) < 13g^4 \cdot h_F(C) + 28g^6 \cdot \log 2.$$

*Proof.* The intersection number $\frac{1}{d}(D' - D, P)$ is bounded by

$$\tfrac{1}{2}h_F(C) - \tfrac{1}{2d}(D, D - \omega_{\mathscr{C}/B}) + 2g^2\Delta(C) + \tfrac{1}{d} \sum_{\sigma\colon K\to\mathbb{C}} \log \|\theta\|_{\sigma,\sup} + \tfrac{g}{2}\log 2\pi,$$

see [EC11, Theorem 9.2.5]. Here, $\|\theta\|_{\sigma,\sup}$ denotes the supremum of $\|\theta\|$ on $\mathrm{Pic}_{g-1}(C_\sigma)$. Since the intersection product is additive and the adjunction formula yields $(P, P) = -(P, \omega_{\mathscr{C}/B})$, see [Ara74, Theorem 4.1], we get

$$-\tfrac{1}{2d}(D, D - \omega_{\mathscr{C}/B}) \leq \tfrac{g+1}{2} \sum_{j=1}^{g} h(W_j) - \tfrac{1}{2d} \sum_{\substack{1\leq j,k\leq g \\ W_j\neq W_k}} (W_j, W_k).$$

We can bound the terms $h(W_j)$ by Proposition 6.1.2 and for $W_j \neq W_k$ we have $-(W_j, W_k) \leq \sum_{\sigma\colon K\to\mathbb{C}} g_\sigma(W_j, W_k)$ with $g_\sigma$ the logarithm of the Arakelov–Green function of $C_\sigma$ for any embedding $\sigma\colon K \to \mathbb{C}$. Using Corollary 4.6.2 we can bound the Arakelov–Green functions in terms of $\delta(C)$, which we can again bound in terms of $h_F(C)$ by the arithmetic Noether formula (6.1.1) and the bound $e(C) \geq 0$. For $g \leq 5$ this yields

$$-\tfrac{1}{2d}(D, D - \omega_{\mathscr{C}/B}) < (3g^4 + 5g^3 + 3g^2 + \tfrac{5}{2}g - \tfrac{3}{2})h_F(C) + 10g^6 \log 2$$

and for $g > 5$

$$-\tfrac{1}{2d}(D, D - \omega_{\mathscr{C}/B}) < (3g^4 + 5g^3 + \tfrac{13}{4}g^2 + \tfrac{7}{8}g - \tfrac{1}{8})h_F(C) + \tfrac{15}{2}g^6 \log 2.$$

Since we have $h_F(C) > -\frac{g}{2}\log 2\pi^2$ by Corollary 6.1.1, we can bound for all $g \geq 2$

$$-\tfrac{1}{2d}(D, D - \omega_{\mathscr{C}/B}) < \tfrac{105}{16}g^4 \cdot h_F(C) + 14g^6 \log 2.$$

Next, we bound $2g^2\Delta(C)$ and $\frac{1}{d}\sum_{\sigma\colon K\to\mathbb{C}} \log\|\theta\|_{\sigma,\sup}$. We apply Lemma 4.5.1 with $r = 1/(2g)$ to bound the supremum $\|\theta\|_{\sigma,\sup}$:

$$\tfrac{1}{d}\sum_{\sigma\colon K\to\mathbb{C}} \log\|\theta\|_{\sigma,\sup} < \left(\tfrac{7}{4}g^3 + \tfrac{9}{8}g^2 + \tfrac{1}{8}g\right)\log 2 - \tfrac{1}{2g}H(C).$$

By (6.1.1), we have $\Delta(C) \leq 12h_F(C) - \delta(C) + 4g\log 2\pi$. If we substitute $\delta(C)$ by Theorem 4.4.1 and if we use the bounds (1.2.9) and Proposition 1.1.1, we conclude

$$2g^2\Delta(C) + \tfrac{1}{d}\sum_{\sigma\colon K\to\mathbb{C}} \log\|\theta\|_{\sigma,\sup} < 24g^2 h_F(C) + 55g^3 \log 2.$$

If we join these bounds together, we obtain the bound in the proposition. □

We can also apply our lower bound for $\delta(C)$ to a result by Javanpeykar and von Känel on Szpiro's small points conjecture. We denote by $S$ the set of places of $K$, where $C$ has bad reduction. Further, we write $D_K$ for the absolute value of the discriminant of $K$ over $\mathbb{Q}$ and we set $N_S = \prod_{v\in S} N_v$ and $\nu = d(5g)^5$. We say that $C$ is a cyclic cover of prime degree if there exists a finite morphism $C \to \mathbb{P}^1_K$ of prime degree, which is geometrically a cyclic cover. By Javanpeykar and von Känel [JvK14, Proposition 5.3] there exist infinitely many geometric points $P$ of $C$, such that

$$h(P) \leq \nu^{8^g d\nu}(N_S D_K)^{\nu} - \min_{X\in\mathcal{M}_g} \delta(X). \tag{6.1.3}$$

Hence, we can apply Corollary 4.4.2 to considerably improve the result in [JvK14, Theorem 3.1] on Szpiro's small points conjecture.

**Corollary 6.1.4.** *Suppose that $C$ is a cyclic cover of prime degree. There are infinitely many geometric points $P \in C(\overline{K})$, which satisfy*

$$\max(h_{NT}(P), h(P)) \leq \nu^{8^g d\nu}(N_S D_K)^{\nu}.$$

*Proof.* This directly follows from (6.1.3) and Corollary 4.4.2. We remark, that the estimates in [JvK14] are coarse enough that we can omit the summand $2g \log 2\pi^4$ resulting from Corollary 4.4.2 and the factor $2g(g-1)$ resulting from (6.1.2). □

## 6.2 Explicit Arakelov theory for hyperelliptic curves

In this section we consider Arakelov theory on hyperelliptic curves. In this special case, we find an explicit description for the stable Arakelov self-intersection number of the relative dualizing sheaf. As applications we obtain an effective version of the Bogomolov conjecture and an arithmetic analogous of the Bogomolov-Miyaoka-Yau inequality. We continue the notation from the last section.

We say $C$ is hyperelliptic if it is in addition given by the projective closure of an equation as in (1.3.1). From now on we assume $C$ to be hyperelliptic. Hence, we can define $\|\Delta_g\|(C) = \frac{1}{d}\sum_\sigma \|\Delta_g\|(C_\sigma)$. Next, we define the type and the subtype of a node. Let $v \in |B|$, such that $C$ has bad reduction at $v$. Choose a node $P$ of the geometric fibre $\mathscr{C}_{\bar{v}}$ and write $(\mathscr{C}_{\bar{v}})_P \to \mathscr{C}_{\bar{v}}$ for the partial normalization at $P$. If $(\mathscr{C}_{\bar{v}})_P$ is connected, we say that $P$ is of type 0. Otherwise, $(\mathscr{C}_{\bar{v}})_P$ has two connected components of arithmetic genus $g_1$ and $g_2$. We may assume $g_1 \le g_2$ and we say that $P$ is of type $g_1$. Since $g_1 + g_2 = g$, we have $g_1 \le \lfloor g/2 \rfloor$. We write $\delta_j(\mathscr{C}_{\bar{v}})$ for the number of all nodes of type $j$ in the geometric fibre $\mathscr{C}_{\bar{v}}$ and we set

$$\Delta_j(C) = \frac{1}{d} \sum_{v \in |B|} \delta_j(\mathscr{C}_{\bar{v}}) \log N_v.$$

It follows $\sum_{j=0}^{\lfloor g/2 \rfloor} \Delta_j(C) = \Delta(C)$.

If $P$ is of type 0, we also define its subtype. The hyperelliptic involution $\sigma$ extends to $\mathscr{C}$, and we denote its restriction to $\mathscr{C}_{\bar{v}}$ by $\sigma_v$. If $\sigma_v(P) = P$ we say $P$ is of subtype 0. Otherwise, the partial normalization $(\mathscr{C}_{\bar{v}})_{P,\sigma_v(P)}$ at $P$ and $\sigma_v(P)$ has two connected components of arithmetic genus $g_1$ and $g_2$. We again assume $g_1 \le g_2$ and say that $P$ is of subtype $g_1$. Since $g_1 + g_2 = g - 1$, we have $g_1 \le \lfloor (g-1)/2 \rfloor$. We write $\xi_0$ for the number of all nodes of subtype 0 in the geometric fibre $\mathscr{C}_{\bar{v}}$. Note that this can also include nodes with $\sigma_v(P) \ne P$ if $\mathscr{C}_{\bar{v}}$ is not stable. For $j \ge 1$ we write $\xi_j(\mathscr{C}_{\bar{v}})$ for the number of $\sigma_v$-orbits of nodes of subtype $j$ in the geometric fibre $\mathscr{C}_{\bar{v}}$. By construction we have

$$\delta_0(\mathscr{C}_{\bar{v}}) = \xi_0(\mathscr{C}_{\bar{v}}) + \sum_{j=1}^{\lfloor \frac{g-1}{2} \rfloor} 2\xi_j(\mathscr{C}_{\bar{v}}).$$

Further, we set

$$\Xi_j(C) = \tfrac{1}{d} \sum_{v \in |B|} \xi_j(\mathscr{C}_{\bar{v}}) \log N_v.$$

Now we can apply our results to the work by Kausz [Kau99] and Yamaki [Yam04] to obtain the following expression for $e(C)$.

**Corollary 6.2.1.** *Let $C$ be any hyperelliptic curve as above. The Arakelov self-intersection number $e(C)$ is given by*

$$\begin{aligned} e(C) =& \tfrac{g-1}{2g+1}\Xi_0(C) + \sum_{j=1}^{\lfloor \frac{g-1}{2} \rfloor} \tfrac{6j(g-1-j)+2(g-1)}{2g+1}\Xi_j(C) \\ &+ \sum_{j=1}^{\lfloor \frac{g}{2} \rfloor} \left( \tfrac{12j(g-j)}{2g+1} - 1 \right) \Delta_j(C) + \tfrac{2(g-1)}{2g+1}\varphi(C). \end{aligned}$$

*Proof.* Kausz constructed a canonical section $\Lambda$ of the metrized line bundle $(\det p_* \omega_{\mathscr{C}/B})^{\otimes(8g+4)}$ in [Kau99, Section 2]. Hence, we can write

$$\widehat{\deg}\left( (\det p_* \omega_{\mathscr{C}/B})^{\otimes(8g+4)} \right) = \sum_{v \in |B|} \mathrm{ord}_v(\Lambda) \log N_v - \sum_{\sigma\colon K \to \mathbb{C}} \log \|\Lambda\|_\sigma.$$

Furthermore, he proved [Kau99, Theorem 3.1], that we have for $v \nmid 2$

$$\mathrm{ord}_v(\Lambda) = g \cdot \xi_0(\mathscr{C}_{\bar{v}}) + 2 \sum_{j=1}^{\lfloor \frac{g-1}{2} \rfloor} (g-j)(j+1)\xi_j(\mathscr{C}_{\bar{v}}) + 4 \sum_{j=1}^{\lfloor \frac{g}{2} \rfloor} (g-j)j\delta_j(\mathscr{C}_{\bar{v}}). \tag{6.2.1}$$

By [Yam04, Theorem 1.7] this equality holds even if $v \mid 2$. However, Yamaki states his theorem only for stable curves. But, we can define the stable model $\mathscr{C} \to \mathscr{C}'$, where the map is given by contracting all rational components in the special fibres of $\mathscr{C}$ meeting the rest of the special fibre in exactly two points. There is a canonical isomorphism $p_* \omega_{\mathscr{C}/B} \cong p'_* \omega_{\mathscr{C}'/B}$, where $p' \colon \mathscr{C}' \to B$ is the structure morphism, see for example [ACG11, Proposition 10.6.7]. Hence, we get an identification of the sections $\Lambda$ and $\Lambda'$ of $(\det p_* \omega_{\mathscr{C}/B})^{\otimes(8g+4)}$, respectively $(\det p'_* \omega_{\mathscr{C}'/B})^{\otimes(8g+4)}$. Further, the calculation in the proof of [Kau99, Lemma 3.2.(b)] shows, that the right hand side of (6.2.1) is compatible with the contraction map $\mathscr{C} \to \mathscr{C}'$. Hence, (6.2.1) also holds for curves with semi-stable reduction at places with $v \mid 2$.

For the Archimedean part we have by [dJo07, p. 11]

$$\tbinom{2g}{g-1} \log \|\Lambda\|_\sigma = g \log \|\Delta_g\|(C_\sigma) + 4g^2 \tbinom{2g+1}{g+1} \log 2\pi.$$

Putting everything together, we get for the Faltings height

$$h_F(C) = \tfrac{1}{8g+4}\left( g\Xi_0(C) + 2\sum_{j=1}^{\lfloor \frac{g-1}{2} \rfloor} (g-j)(j+1)\Xi_j(C) + 4\sum_{j=1}^{\lfloor \frac{g}{2} \rfloor}(g-j)j\Delta_j(C)\right)$$
$$- \tfrac{g}{8g+4}\tbinom{2g}{g-1}^{-1} \log \|\Delta_g\|(C) - g\log 2\pi.$$

If we apply Corollary 3.3.2 to the arithmetic Noether formula (6.1.1), we get

$$12h_F(C) = e(C) + \Delta(C) - \tfrac{2(g-1)}{2g+1}\varphi(C) - \tfrac{3g}{2g+1}\tbinom{2g}{g-1}^{-1} \log \|\Delta_g\|(C) - 12g\log 2\pi.$$

Now the corollary follows by combining both formulas and solving for $e(C)$. □

This explicit expression for $e(C)$ leads to an effective version of the Bogomolov conjecture for hyperelliptic curves in the same way as Yamaki [Yam08] worked this out for function fields.

**Corollary 6.2.2.** *Let $C$ be any hyperelliptic curve as above and $z$ any geometric point of $Pic^0(C)$. There are only finitely many geometric points $P \in C(\overline{K})$ satisfying*

$$h_{NT}(((2g-2)P - K_C) - z) \leq \tfrac{(g-1)^2}{2g+1}\left(\tfrac{2g-5}{12g}\Delta(C) + \varphi(C)\right).$$

*Proof.* Zhang introduced in [Zha93] the notion of the admissible pairing $(\cdot,\cdot)_a$ and of the admissible dualizing sheaf $\omega^a_{\mathscr{C}/B}$, and he proved, that

$$\lim \inf_{P\in C(\overline{K})} h_{NT}((2g-2)P - K_C) - z) \geq \tfrac{g-1}{d}(\omega^a_{\mathscr{C}/B}, \omega^a_{\mathscr{C}/B})_a,$$

see [Zha93, Theorem 5.6]. The admissible self-intersection number of $\omega^a_{\mathscr{C}/B}$ satisfies

$$(\omega^a_{\mathscr{C}/B}, \omega^a_{\mathscr{C}/B})_a = (\omega_{\mathscr{C}/B}, \omega_{\mathscr{C}/B}) - \sum_{v\in|B|} \epsilon_v \log N_v,$$

where the $\epsilon_v$'s are non-negative constants depending only on the weighted dual graph of $\mathscr{C}_{\bar{v}}$ and we have $\epsilon_v = 0$ if $C$ has good reduction at $v$, see [Zha93, Theorem 5.5]. Hence, we have to bound $\epsilon_v$ in terms of $\xi_j(\mathscr{C}_{\bar{v}})$ and $\delta_k(\mathscr{C}_{\bar{v}})$ for $0 \le j \le \lfloor (g-1)/2 \rfloor$ and $1 \le k \le \lfloor g/2 \rfloor$. This was done by Yamaki [Yam08] for the function field case. Since we are only interested in the weighted dual graph of a special fibre, the calculation is exactly the same. Thus, we obtain for $g \ge 5$

$$\epsilon_v \le \tfrac{5(g-1)}{12g}\xi_0(\mathscr{C}_{\bar{v}}) + \sum_{j=1}^{\lfloor \frac{g-1}{2} \rfloor} \tfrac{4(g-1)+6j(g-1-j)}{3g}\xi_j(\mathscr{C}_{\bar{v}}) + \sum_{j=1}^{\lfloor \frac{g}{2} \rfloor} \left(\tfrac{4j(g-j)}{g} - 1\right)\delta_j(\mathscr{C}_{\bar{v}})$$

and for $g \le 4$

$$\epsilon_v \le \tfrac{5(g-1)}{12g}\xi_0(\mathscr{C}_{\bar{v}}) + \sum_{j=1}^{\lfloor \frac{g-1}{2} \rfloor} \tfrac{g-1+2j(g-1-j)}{g}\xi_j(\mathscr{C}_{\bar{v}}) + \sum_{j=1}^{\lfloor \frac{g}{2} \rfloor} \left(\tfrac{4j(g-j)}{g} - 1\right)\delta_j(\mathscr{C}_{\bar{v}}),$$

see [Yam08, Section 4.3]. Using these bounds, we can estimate by Corollary 6.2.1

$$\tfrac{1}{d}(\omega^a_{\mathscr{C}/B}, \omega^a_{\mathscr{C}/B})_a \ge \tfrac{(g-1)(2g-5)}{12g(2g+1)}\Delta(C) + \tfrac{2(g-1)}{2g+1}\varphi(C).$$

Hence, the corollary follows by $\varphi(C) > 0$. □

By elementary estimations of the coefficients in Corollary 6.2.1, we deduce the following bounds for $e(C)$.

**Corollary 6.2.3.** *Let $C$ be any hyperelliptic curve as above. The Arakelov self-intersection number $e(C)$ is bounded in the following way:*

$$\tfrac{g-1}{2g+1}(\Delta(C) + 2\varphi(C)) \le e(C) \le \tfrac{g-1}{2g+1}((3g+1)\Delta(C) + 2\varphi(C)).$$

As a consequence we deduce an arithmetic analogous of the Bogomolov–Miyaoka–Yau inequality, as suggested by Parshin [Par90, §1.(10)], for hyperelliptic curves.

**Corollary 6.2.4.** *Let $C$ be any hyperelliptic curve as above. We have*

$$e(C) < \tfrac{g-1}{2g+1}\left((3g+1)\Delta(C) + \delta(C) + 2g\log 2\pi^4\right)$$

*and in terms of the more explicit invariant* $\log\|\Delta_g\|(C)$

$$e(C) < \tfrac{g-1}{2g+1}\left((3g+1)\Delta(C) - \tbinom{2g}{g-1}^{-1}\log\|\Delta_g\|(C) - 2(2g+1)\log 2\right).$$

*Proof.* An application of the bound in Proposition 1.1.1 to Corollary 3.3.3 and the lower bound in Corollary 3.3.5 yield

$$2\varphi(C) < -\binom{2g}{g-1}^{-1} \log \|\Delta_g\|(C) - 2(2g+1)\log 2 < \delta(C) + 2g \log 2\pi^4.$$

Now the corollary follows by combining these bounds with the upper bound for $e(C)$ in Corollary 6.2.3. □

A similar but weaker bound was already obtained by Kausz [Kau99, Corollary 7.8] and Maugeais [Mau03, Corollaire 2.11]. However, their bounds involve an additional constant, which is not explicitly given. Parshin observed in [Par90] that a certain upper bound for $(\omega_{V/B}, \omega_{V/B})$ for all arithmetic surfaces $V \to B$ with stable fibres and smooth generic fibre of genus $g \geq 2$ would imply interesting arithmetic consequences, for example the *abc*-conjecture. Unfortunately, we can not deduce any arithmetic consequences from the special case Corollary 6.2.4 by the same methods as in [Par90].

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