The following text is a preprint of the paper:

N. Fries, M. Dreyer; Dimensionless scaling methods for capillary rise; Journal of Colloid and Interface Science 338: 514-518, 2009.

www.elsevier.com/locate/jcis or

http://dx.doi.org/10.1016/j.jcis.2009.06.036

Dimensionless scaling methods for capillary rise

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Abstract

In this article the different dimensionless scaling methods for capillary rise of liquids in a tube or a porous medium are discussed. A systematic approach is taken, and the possible options are derived by means of the Buckingham π theorem. It is found that three forces (inertial, viscous and hydrostatic forces) can be used to obtain three different scaling sets, each consisting of two dimensionless variables and one dimensionless basic parameter. From a general point of view the three scaling options are all equivalent and valid for describing the problem of capillary rise. Contrary to this we find that for certain cases (depending on the time scale and the dominant forces) one of the options can be favorable. Individually the different scalings have been discussed and used in literature previously, however, we intend to discuss the three different sets systematically in a single paper and try to evaluate when which scaling is most useful. Furthermore we investigate previous analytic solutions and determine their ranges of applicability when compared to numerical solutions of the differential equation of motion (momentum balance).

Key words: Dimensionless scaling, Capillary tube, Analytic solution, Capillary rise, Lucas-Washburn equation, Washburn equation, Imbibition, Liquid penetration

1. Introduction

There are numerous applications of capillary transport phenomena ranging from daily life (writing with ink) to complex engineering applications (fluid management in space) and pure academic interest (validation of CFD tools). Thus there are many publications dealing with this problem, its mathematical description and its physical explanation [1–7]. To obtain a better understanding of a problem its dimensionless consideration is always of interest. Here the Buckingham π theorem [8] can be used to obtain appropriate dimensionless scalings. In literature there are several papers applying dimensionless numbers to the problem of capillary rise. Ichikawa and Satoda [9] focus on experiments with horizontal capillaries, Dreyer et al. [10] and Stange et al. [11,12] on capillaries in a microgravity environment. There also exist studies involving gravity, thus leading to different scaling approaches e.g. by Quéré et al. [13,14], Marmur and Cohen [15], Zhmud et al. [16], Lee and Lee [17] or Fries and Dreyer [18,19]. McKinley [20] investigates dimensionless groups for free surface flows with a focus on complex fluids. In this paper we now intend to follow a systematic approach to dimensionless scaling of capillary rise, and to compare the different derived options. The basis for the dimensionless scalings is the differential equation of motion of the liquid inside a capillary tube. It can be derived by solving an integral balance of the linear momentum in an appropriate control volume [5]. To solve the integrals and to obtain the boundary conditions some assumptions have been made. First of all the viscous losses in the tube are described using the Hagen-Poiseuille law. Also the capillary pressure is assumed to be constant, hence a static contact angle θ is used (e.g. see [6,19]). Furthermore entry effects and losses in the liquid reservoir are neglected. With these assumptions the equation of motion is given by (e.g.[3,16])

$$-\rho\frac{d(h\dot{h})}{dt} = -\frac{2\sigma\cos\theta}{R} + \frac{8\mu h}{R^2}\dot{h} + \rho g h \quad (\text{for } \dot{h} > 0). \quad (1)$$

In this equation the momentum change (inertia, left hand side) is balanced by the capillary pressure, the viscous forces and the hydrostatic pressure (left to right). σ refers to the surface tension, R to the inner tube radius, ρ to the fluid density, g to gravity and μ to the fluid viscosity. It is interesting to note that Eq. (1) is only valid for a rising column. For a falling column - as it occurs in oscillating cases - the different flow characteristics at the tube inlet have to be considered. While for the rising column it acts as a sink, a jet is emitted for the falling column. For the descending

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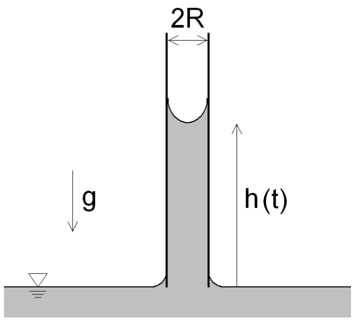

Fig. 1. Liquid rise in a capillary tube [19]

case, a $\dot{h}^2$ term included in the left hand side of Eq. (1) has to be omitted to obtain

$$-\rho h\ddot{h} = -\frac{2\sigma\cos\theta}{R} + \frac{8\mu h}{R^2}\dot{h} + \rho g h \quad (\text{for } \dot{h} < 0) \qquad (2)$$

as shown by Lorenceau et al. [21].
The momentum balance can also be given for the capillary rise of liquids in porous media, here the viscous term is replaced by the Darcy law

$$-\rho\frac{d(h\dot{h})}{dt} = -\frac{2\sigma\cos\theta}{R} + \frac{\phi\mu h}{K}\dot{h} + \rho g h \quad (\text{for } \dot{h} > 0). \qquad (3)$$

ϕ denotes the porosity of the structure, and K its permeability.

2. Dimensionless scaling

In this section the different dimensionless scaling options will be discussed. The Buckingham π theorem and the approach described by White [22] is used. The relevant definitions shall be introduced briefly:

• Dimensional variables are the basic output of the experiment, and normally the ones to be shown in a diagram. They vary during a given run. In our case h and t (see Fig. 1).
• Dimensional parameters affect the variables, and may vary from case to case, however remain constant during a given run. In our case a, b and c, see Eqs. (4), (5) and (6) below.
• Fundamental units are the units of the variables and parameters e.g. meter, kilogram, second.
• Scaling parameters are chosen to convert the variables to a dimensionless form. In our case: two can be chosen.
• Basic parameter is the - in our case one - remaining parameter.
• Dimensionless variables are the variables made dimensionless by the scaling parameters.
• Dimensionless basic parameter is the basic parameter made dimensionless using the scaling parameters.

In a graphic representation of the dimensionless solution the axes are the dimensionless variables, while the dimensionless basic parameter is varied to plot a set of curves [22] (e.g. see Fig. 2). With varying dimensionless basic parameter the influence of the basic parameter (and the corresponding force) can be observed. Regarding Eqs. (1) and (3) we may define the following dimensional parameters

$$a = \frac{\rho R}{2\sigma\cos\theta}, \qquad (4)$$

$$b = \frac{4\mu}{R\sigma\cos\theta} \mathrel{\widehat{=}} \frac{\phi\mu R}{2K\sigma\cos\theta}, \qquad (5)$$

$$c = \frac{\rho g R}{2\sigma\cos\theta}. \qquad (6)$$

For b both the capillary tube and the Darcy version is given. However, in favor of readability, we will not continue to explicate the Darcy version in the further text. Please note that the parameters a, b and c are not identical to those applied in [18,19]. Using the introduced dimensional parameters one can rearrange Eqs. (1) and (3) to obtain

$$\underbrace{a\frac{d\,(h\dot{h})}{dt}}_{inertial} + \underbrace{bh\dot{h}}_{viscous} + \underbrace{ch}_{hydrostatic} = 1. \qquad (7)$$

It can now be observed that the momentum balance has become much more clearly arranged and that each dimensional parameter stands for a single term: a - inertia, b - viscous effects and c - hydrostatic effects. Table 1 summarizes the three different scaling options that will be examined one by one in the next sections.

Table 1
Scaling options

Option	Basic parameter	Scaling parameters
1	a (inertia)	b (viscosity) and c (gravity)
2	b (viscosity)	a (inertia) and c (gravity)
3	c (gravity)	a (inertia) and b (viscosity)

3. Viscous effects and gravity as scaling forces (†)

Here, b (viscous effects) and c (gravity) are used as scaling parameters, the remaining parameter a (inertia) is used as basic parameter. The resulting dimensionless variables and the dimensionless basic parameter are derived by applying the Buckingham π theorem as shown in the Appendix

$$\pi_1^\dagger = h^\dagger = ch = \frac{\rho g R}{2\sigma\cos\theta}h, \qquad (8)$$

and

$$\pi_2^\dagger = t^\dagger = \frac{c^2 t}{b} = \frac{\rho^2 g^2 R^3}{16\mu\sigma\cos\theta}t. \qquad (9)$$

These two dimensionless variables have been used by Zhmud et al. [16] and Fries and Dreyer [18]. The dimensionless basic parameter reads as follows

$$\pi_3^\dagger = \Omega = \sqrt{\frac{b^2}{ac^2}} = \sqrt{\frac{128\sigma\cos\theta\mu^2}{\rho^3 g^2 R^5}}. \qquad (10)$$

According to Quéré et al. [14], we denote the basic dimensionless parameter $\pi_3^\dagger$ as Ω. Here, Ω can be used to measure the influence of inertia. In Fig. 2 it can be seen that for decreasing Ω (increasing inertia, see arrow) the

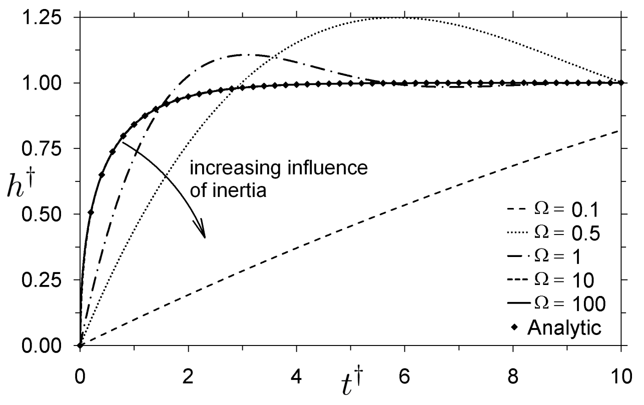

Fig. 2. Plot showing the dimensionless numerical solution of Eqs. (11) and (12). Viscosity and gravity are the scaling forces, inertia is the basic parameter for the set of curves. The points refer to the analytic solution for $\Omega \to \infty$ by Washburn.

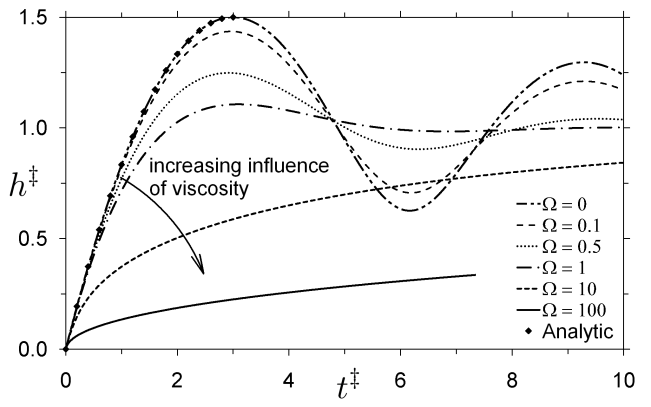

Fig. 3. Plot showing the dimensionless numerical solution of Eqs. (18) and (19). Inertia and gravity are the scaling forces, viscosity is the basic parameter for the set of curves. The points refer to the analytic solution for $\Omega \to 0$ by Quéré.

oscillations and the overshoot increase. This is consistent with Quéré et al. who find oscillations to occur for $\Omega \leq 2$. It is interesting to note that for all three scaling options presented in this article Ω ($=\pi_3$) is mathematically the same, however, its meaning changes from scaling to scaling [22]. Thus Ω always reflects the influence of the chosen basic parameter. For example, as will be shown later in further detail, Ω can become infinite in two limits which are physically very different: For a non inertial case (the Washburn limit) with $a = 0$, and for the no gravity case (the Bosanquet limit) with $c = 0$.
The numerical solutions of the momentum balance as shown in Figs. 2, 3 and 4 have been obtained by using an implicit Runge-Kutta algorithm with the initial conditions $h(t = 0) = 0$ and $\dot{h}(t = 0) = 0$. The case differentiation for $\dot{h} > 0$ and $\dot{h} < 0$ was programmed by including an if() command into the code.
Applying the scalings presented above the resulting dimensionless momentum balances read

$$\frac{1}{\Omega^2}\frac{d\left(h^\dagger \frac{dh^\dagger}{dt^\dagger}\right)}{dt^\dagger} + h^\dagger \frac{dh^\dagger}{dt^\dagger} + h^\dagger = 1 \quad (\text{for } \dot{h}^\dagger > 0) \tag{11}$$

and

$$\frac{1}{\Omega^2}\frac{d^2 h^\dagger}{dt^{\dagger\,2}} + h^\dagger \frac{dh^\dagger}{dt^\dagger} + h^\dagger = 1 \quad (\text{for } \dot{h}^\dagger < 0). \tag{12}$$

For $\Omega \to \infty$ (no inertia), Eq. (11) can be solved analytically with the solution given in implicit form by Washburn [2]

$$t^\dagger = -h^\dagger - \ln(1 - h^\dagger), \tag{13}$$

and in explicit form by Barry et al. [23] and Fries and Dreyer [18]

$$h^\dagger = 1 + W(-e^{-1-t^\dagger}). \tag{14}$$

Hereby, $W(x)$ denotes the Lambert W function. By numerical means we now find that the deviation between the analytic and the numerical solution is smaller than 5 % for $\Omega \geq 7.9$ and $t^\dagger \geq 0.1$.

4. Inertia and gravity as scaling forces (‡)

For this case one obtains - analogous to the procedure as shown in the Appendix - the following dimensionless variables

$$\pi_1^\ddagger = h^\ddagger = ch = \frac{\rho g R}{2\sigma\cos\theta}h, \tag{15}$$

and

$$\pi_2^\ddagger = t^\ddagger = \sqrt{\frac{c^2}{a}}\,t = \sqrt{\frac{\rho g^2 R}{2\sigma\cos\theta}}\,t. \tag{16}$$

These dimensionless variables have been used by Quéré et al. [14]. The dimensionless basic parameter reads

$$\pi_3^\ddagger = \Omega = \sqrt{\frac{b^2}{ac^2}} = \sqrt{\frac{128\sigma\cos\theta\mu^2}{\rho^3 g^2 R^5}}. \tag{17}$$

Here, Ω can be used to measure the influence of viscous effects. Thus it can be observed that in Fig. 3 the oscillations decrease with increasing Ω (increasing viscosity, see arrow). The dimensionless momentum balances read

$$\frac{d\left(h^\ddagger \frac{dh^\ddagger}{dt^\ddagger}\right)}{dt^\ddagger} + \Omega h^\ddagger \frac{dh^\ddagger}{dt^\ddagger} + h^\ddagger = 1 \quad (\text{for } \dot{h}^\ddagger > 0) \tag{18}$$

and

$$\frac{d^2 h^\ddagger}{dt^{\ddagger\,2}} + \Omega h^\ddagger \frac{dh^\ddagger}{dt^\ddagger} + h^\ddagger = 1 \quad (\text{for } \dot{h}^\ddagger < 0). \tag{19}$$

For $\Omega \to 0$ (no viscous effects), Eq. (18) can be solved analytically with the solution given by Quéré [13] to be

$$h^\ddagger = t^\ddagger\left(1 - \frac{t^\ddagger}{6}\right), \tag{20}$$

valid for $0 \leq t^\ddagger \leq 3$. By numerical means we now find that the deviation between the analytic and the numerical solution is smaller than 5 % for $\Omega \leq 0.11$ and $0 \leq t^\ddagger \leq 3$.

5. Inertia and viscous effects as scaling forces (∗)

With this choice one obtains - analogous to the procedure as shown in the Appendix - the dimensionless variables as described in following: The first one reads

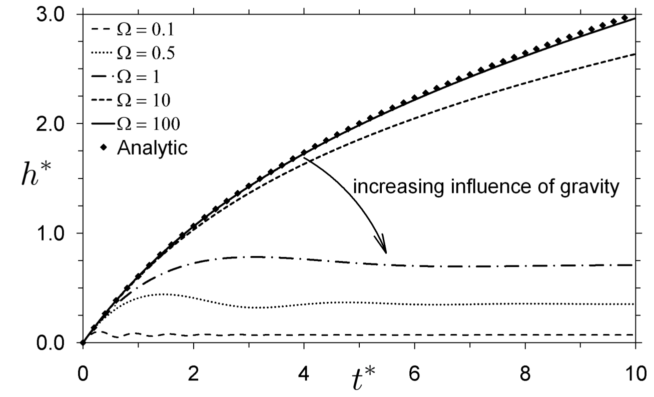

Fig. 4. Plot showing the dimensionless numerical solution of Eqs. (25) and (26). Inertia and viscosity are the scaling forces, gravity is the basic parameter for the set of curves. The points refer to the analytic solution for $\Omega \to \infty$ by Bosanquet.

$$\pi_1^* = \sqrt{\frac{b^2}{a}}\, h = \sqrt{\frac{32\mu^2}{\rho R^3 \sigma \cos\theta}}\, h. \tag{21}$$

To be consistent with the scaling by Ichikawa and Satoda [9], and to obtain a more elegant form of analytic solution (see Eq. (27)) we divide π_1^* by $\sqrt{2}$ to obtain

$$h^* = \frac{b}{\sqrt{2a}}\, h = \sqrt{\frac{16\mu^2}{\rho R^3 \sigma \cos\theta}}\, h. \tag{22}$$

The second dimensionless variable reads

$$\pi_2^* = t^* = \frac{b}{a} t = \frac{8\mu}{\rho R^2} t, \tag{23}$$

and the dimensionless basic parameter

$$\pi_3^* = \Omega = \sqrt{\frac{b^2}{ac^2}} = \sqrt{\frac{128\sigma\cos\theta\mu^2}{\rho^3 g^2 R^5}}. \tag{24}$$

Here, Ω can be used to measure the influence of hydrostatic effects. Thus it can be observed that in Fig. 4 the oscillations increase and the maximum height decreases with decreasing Ω (increasing hydrostatic effect, see arrow). Using the presented scaling the dimensionless momentum balances can be given as

$$2\frac{d\left(h^* \frac{dh^*}{dt^*}\right)}{dt^*} + 2h^* \frac{dh^*}{dt^*} + \frac{\sqrt{2}}{\Omega} h^* = 1 \quad (\text{for } \dot{h}^* > 0) \tag{25}$$

and

$$2\frac{d^2 h^*}{dt^{*\,2}} + 2h^* \frac{dh^*}{dt^*} + \frac{\sqrt{2}}{\Omega} h^* = 1 \quad (\text{for } \dot{h}^* < 0). \tag{26}$$

For $\Omega \to \infty$ (no hydrostatic effects), Eq. (25) can be solved analytically with the solution given by Bosanquet [3] to be

$$h^* = \sqrt{t^* - (1 - e^{-t^*})}. \tag{27}$$

By numerical means we now find that the deviation between the analytic and the numerical solution is smaller than 5 % for $\Omega \geq 96$ and $t^* \leq 100$.

Table 2
Overview of dimensionless variables and Ω. See Tab. 3 for description of dimensionless numbers.

Variable	Dimensionless numbers	$\sim$ Forces
$h^\dagger$	$\sim \frac{\mathrm{Bo}}{\cos\theta}$	$\frac{\text{gravity}}{\text{surface tension}}$
$t^\dagger$	$\sim \frac{\mathrm{Bo}^2}{\mathrm{Ca}\,\cos\theta}$	$\frac{(\text{gravity})^2}{\text{viscous} \cdot \text{surface tension}}$
$h^\ddagger$	$\sim \frac{\mathrm{Bo}}{\cos\theta}$	$\frac{\text{gravity}}{\text{surface tension}}$
$t^\ddagger$	$\sim \sqrt{\frac{\mathrm{Bo}}{\mathrm{Fr}^2\cos\theta}}$	$\sqrt{\frac{(\text{gravity})^3}{(\text{inertia})^2 \cdot \text{surface tension}}}$
h^*	$\sim \frac{\mathrm{Oh}}{\sqrt{\cos\theta}}$	$\frac{\text{viscous}}{\sqrt{\text{inertia} \cdot \text{surface tension}}}$
t^*	$\sim \frac{1}{\mathrm{Re}}$	$\frac{\text{viscous}}{\text{inertia}}$
Ω	$\sim \sqrt{\frac{\cos\theta}{\mathrm{Bo}\cdot\mathrm{Ga}}}$	$\frac{\sqrt{\text{surface tension} \cdot \text{viscous}}}{\text{gravity}}$

6. Discussion

In Table 2 the different dimensionless variables and Ω are examined further on. It can be observed that they are related to the indicated forces and well known dimensionless numbers as displayed in Table 3. In Table 2, $\cos\theta$ appears as an independent dimensionless parameter.

It is interesting to note that the three figures (Figs. 2, 3, 4) shown in the previous sections all represent solutions of the same five cases (equal Ω). Due to the different scalings however, their shapes bear no direct resemblance. From a general point of view, the three scaling options are all equivalent for describing the problem of capillary rise. However, for some cases there can be a benefit of choosing a certain scaling method. In the following, two of these cases shall be discussed:

Table 3
List of relevant dimensionless numbers

Abbrev.	Name	Equation	Forces
Bo	Bond-number	$\frac{\rho g R^2}{\sigma}$	$\frac{\text{gravity}}{\text{surface tension}}$
Ga	Galileo-number	$\frac{g R^3 \rho^2}{\mu^2}$	$\frac{\text{gravity}}{\text{viscous}}$
Ca	Capillary-number	$\frac{\mu v}{\sigma} \sim \frac{\mu R}{\sigma t}$	$\frac{\text{viscous}}{\text{surface tension}}$
Oh	Ohnesorge-number	$\frac{\mu}{\sqrt{R\rho\sigma}}$	$\frac{\text{viscous}}{\sqrt{\text{inertia} \cdot \text{surface tension}}}$
Re	Reynolds-number	$\frac{\rho R v}{\mu} \sim \frac{\rho R^2}{\mu t}$	$\frac{\text{inertia}}{\text{viscous}}$
Fr	Froude-number	$\frac{v}{\sqrt{gR}} \sim \sqrt{\frac{R}{g t^2}}$	$\frac{\text{inertia}}{\text{gravity}}$

All forces are effective

One can visualize the impact of a parameter that is to be investigated by choosing it to be the basic parameter. Then the two remaining parameters must act as scaling parameters. If, for example, viscous effects and gravity are chosen to be scaling forces (parameters), then Ω, the basic parameter for the set of curves, will reflect the influence of inertia.

One of the forces can be neglected

For some cases it is possible to neglect the influence of a certain force, e.g. in microgravity the hydrostatic term can be neglected. This also applies for experiments where only the initial time period of the capillary rise is investigated, while for later time stages in small capillaries one can usually neglect inertia. The neglected force can not be used as a scaling force, thus the one remaining scaling option should be chosen. In this case the neglected force will act as basic parameter for the set of curves. However, the plotted solution is reduced to a single curve as the basic dimensionless parameter Ω will equal 0 or ∞.

7. Conclusion

In this paper we have demonstrated that one can use the Buckingham π theorem to systematically derive three different scaling options. Each option consists of a set of two dimensionless variables and one basic dimensionless parameter. The different options found are discussed and numerical as well as analytic solutions of the momentum balance are shown in dimensionless form. Generally, the different scaling options are absolutely equivalent in terms of describing the problem. However, using the right scaling can help to identify the influence of a certain parameter to be investigated. Also for some special cases (e.g. microgravity) the choice is limited to a single scaling method. These findings can help to choose an appropriate scaling for representing experimental data of capillary rise, and they may also help to systematically plan an experimental campaign well in advance by defining which dimensionless basic parameter is to be varied.

Appendix - Applying the Buckingham π Theorem

The following appendix aims to clarify the procedure to derive the scalings. For the problem discussed we find five (5) dimensional units:
a $[\frac{\mathrm{s}^2}{\mathrm{m}^2}]$
b $[\frac{\mathrm{s}}{\mathrm{m}^2}]$
c $[\frac{1}{\mathrm{m}}]$
h [m]
t [s]

and two (2) fundamental units:
time [s]
length [m]

Thus one will obtain $5-2=3$ dimensionless π parameters that characterize the problem. The table of fundamental units reads as given in Tab. 4. According to the Buckingham π theorem the system of equations evolves as follows

Table 4
Fundamental units

-	a	b	c	h	t
seconds	2	1	0	0	1
meters	-2	-2	-1	1	0

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 1 \\ -2 & -2 & -1 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \pi_{1,a} & \pi_{2,a} & \pi_{3,a} \\ \pi_{1,b} & \pi_{2,b} & \pi_{3,b} \\ \pi_{1,c} & \pi_{2,c} & \pi_{3,c} \\ \pi_{1,h} & \pi_{2,h} & \pi_{3,h} \\ \pi_{1,t} & \pi_{2,t} & \pi_{3,t} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{28}$$

This system is under-determined, and one is allowed to choose three parameters in each π vector.

Viscous effects and gravity as scaling forces (†)

As we have chosen that a (inertia) shall be the basic parameter and b (viscosity) and c (gravity) shall be the scaling parameters we set for an appropriate scaling for h that $\pi_{1,a}=0$, $\pi_{1,h}=1$, and $\pi_{1,t}=0$. To find a parameter for t we set $\pi_{2,a}=0$, $\pi_{2,h}=0$, and $\pi_{2,t}=1$. To find the basic dimensionless parameter we use $\pi_{3,a}=-1/2$ (this is chosen to be consistent with Ω as defined by Quéré et al., other choices lead to linearly dependent solutions), $\pi_{3,h}=0$, and $\pi_{3,t}=0$. Now one can solve to obtain:

$$\begin{pmatrix} \pi_{1,a} & \pi_{2,a} & \pi_{3,a} \\ \pi_{1,b} & \pi_{2,b} & \pi_{3,b} \\ \pi_{1,c} & \pi_{2,c} & \pi_{3,c} \\ \pi_{1,h} & \pi_{2,h} & \pi_{3,h} \\ \pi_{1,t} & \pi_{2,t} & \pi_{3,t} \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1/2 \\ 0 & -1 & 1 \\ 1 & 2 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{29}$$

Thus the Buckingham π theorem provides us with two dimensionless variables and one dimensionless basic parameter.

$$\pi_1^\dagger = a^0 b^0 c^1 h^1 t^0 \tag{30}$$

$$\pi_2^\dagger = a^0 b^{-1} c^2 h^0 t^1 \tag{31}$$

$$\pi_3^\dagger = a^{-\frac{1}{2}} b^1 c^{-1} h^0 t^0 \tag{32}$$

Inertia and gravity as scaling forces (‡)

As we have chosen that b (viscosity) shall be the basic parameter and a (inertia) and c (gravity) shall be the scaling parameters we set for an appropriate scaling for h that $\pi_{1,b}=0$, $\pi_{1,h}=1$, and $\pi_{1,t}=0$. To find a parameter for t we set $\pi_{2,b}=0$, $\pi_{2,h}=0$, and $\pi_{2,t}=1$. To find the basic dimensionless parameter we use $\pi_{3,b}=1$, $\pi_{3,h}=0$, and $\pi_{3,t}=0$. Now one can solve to obtain:

$$\begin{pmatrix} \pi_{1,a} & \pi_{2,a} & \pi_{3,a} \\ \pi_{1,b} & \pi_{2,b} & \pi_{3,b} \\ \pi_{1,c} & \pi_{2,c} & \pi_{3,c} \\ \pi_{1,h} & \pi_{2,h} & \pi_{3,h} \\ \pi_{1,t} & \pi_{2,t} & \pi_{3,t} \end{pmatrix} = \begin{pmatrix} 0 & -1/2 & -1/2 \\ 0 & 0 & 1 \\ 1 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{33}$$

Thus the Buckingham π theorem provides us with two dimensionless variables and one dimensionless basic parameter.

$$\pi_1^\ddagger = a^0 b^0 c^1 h^1 t^0 \tag{34}$$

$$\pi_2^\ddagger = a^{-\frac{1}{2}} b^0 c^1 h^0 t^1 \tag{35}$$

$$\pi_3^\ddagger = a^{-\frac{1}{2}} b^1 c^{-1} h^0 t^0 \tag{36}$$

Inertia and viscous effects as scaling forces ($*$)

As we have chosen that c (gravity) shall be the basic parameter and a (inertia) and b (viscosity) shall be the scaling parameters we set for an appropriate scaling for h that $\pi_{1,c}=0$, $\pi_{1,h}=1$, and $\pi_{1,t}=0$. To find a parameter for t

we set $\pi_{2,c} = 0$, $\pi_{2,h} = 0$, and $\pi_{2,t} = 1$. To find the basic dimensionless parameter we use $\pi_{3,c} = -1$ (this is chosen to be consistent with Ω as defined by Quéré et al., other choices lead to linearly dependent solutions), $\pi_{3,h} = 0$, and $\pi_{3,t} = 0$. Now one can solve to obtain:

$$\begin{pmatrix} \pi_{1,a} & \pi_{2,a} & \pi_{3,a} \\ \pi_{1,b} & \pi_{2,b} & \pi_{3,b} \\ \pi_{1,c} & \pi_{2,c} & \pi_{3,c} \\ \pi_{1,h} & \pi_{2,h} & \pi_{3,h} \\ \pi_{1,t} & \pi_{2,t} & \pi_{3,t} \end{pmatrix} = \begin{pmatrix} -1/2 & -1 & -1/2 \\ 1 & 1 & 1 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{37}$$

Thus the Buckingham π theorem provides us with two dimensionless variables and one dimensionless basic parameter.

$$\pi_1^* = a^{-\frac{1}{2}} b^1 c^0 h^1 t^0 \tag{38}$$

$$\pi_2^* = a^{-1} b^1 c^0 h^0 t^1 \tag{39}$$

$$\pi_3^* = a^{-\frac{1}{2}} b^1 c^{-1} h^0 t^0 \tag{40}$$

Acknowledgement

This work was supported by the German Research Foundation (DFG) through the Research Training Group 1375.

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