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Efficient constraint adaptation for sampled linear systems

Moritz Schulze Darup[†]

Abstract

The paper addresses rigorous constraint satisfaction for linear continuous-time systems under piecewise constant control. To guarantee constraint satisfaction, we discretize the system and design adapted constraints in such a way that constraint satisfaction of the discretized system w.r.t. the adapted constraints implies constraint satisfaction of the continuous-time system w.r.t. the original constraints. Compared to existing approaches, the new method described in this paper is less conservative, not restricted to any specific control scheme (e.g., MPC), and computationally efficient.

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1 Introduction

In the majority of cases, mathematical models describing real-life processes are continuous-time systems. In contrast, due to sampling, controller design and evaluation are usually carried out in the discrete-time domain. To this end, the process model is often transformed into a discrete-time system. In principle, this procedure is uncritical for linear systems since the discretization is exact in the sense that the discretized system and the continuous-time system (under zero-order hold (ZOH) control) coincide at all sampling instances.

The situation changes, however, if state and input constraints are present. In fact, even for constrained *linear* systems, it is well-known that the continuous-time system (i.e., the real-life process) may violate some constraints although the discrete-time counterpart satisfies all constraints for all sampling instances

[†] M. Schulze Darup is with the Control Group, Department of Engineering Science, University of Oxford, Parks Road, Oxford OX1 3PJ, UK. E-mail: moritz.schulzedarup@rub.de.

(see, e.g., the motivating example in [12]). An obvious way to avoid this problem is to adapt the constraints such that constraint satisfaction of the discrete-time systems (w.r.t. the adapted constraints) implies constraint satisfaction of the continuous-time system (w.r.t. the original constraints).

The paper deals with a new method for the systematic adaptation of the original constraints of the continuous-time system. To highlight differences to existing procedures, we briefly summarize the results obtained in [1, 2, 7, 12] and discuss advantages of the new approach afterwards.

In [1] and [2], methods for the computation of so-called *controlled safe sets* are presented. Controlled safe sets are related to but different from controlled invariant sets. According to [2, Def. 2], for every state in a Δt-controlled safe set, there exists a piecewise constant (or ZOH) control law such that the controlled continuous-time system respects the original constraints for all times. While the computation of controlled safe sets is well-understood, it is not yet clear how to integrate the computed sets into common control schemes. In fact, staying in the controlled safe set only guarantees that suitable control actions *exist*. However, the controller has to take care that one of these suitable controls is indeed *selected*, which is not straightforward.

In [12] and [7], methods guaranteeing constraint satisfaction of continuous-time systems under sampled model predictive control (MPC) are addressed. While the reformulated MPC schemes indeed preclude violation of the original constraints, both approaches concentrate on a specific control strategy, namely MPC.

In contrast to [1] and [2], the new method presented in this paper provides adapted constraints that do not require the verification of suitable inputs during runtime of the controller. In other words, *if* the discretized system satisfies the adapted constraints provided here, *then* the continuous-time system is guaranteed to respect the original constraints. Moreover, in contrast to [12] and [7], the presented method is not restricted to any specific control scheme. In fact, it can be combined with any control strategy that takes constraints explicitly into account (like but not limited to MPC). Furthermore, we consider combined state and input constraints and thus generalize the concepts introduced in [1, 2, 7, 12]. Finally, the adapted constraints computed by the proposed procedure can be significantly less conservative than those obtained using existing approaches (see the example in Sect. 4).

The paper is organized as follows. After introducing some notation in the remainder of this section, we detail the problem of interest in Section 2. The main results of the paper, i.e., the efficient computation of adapted constraints is addressed in Section 3. Finally, Sections 4 and 5 present a numerical example and state conclusions, respectively.

1.1 Notation

We denote non-negative reals, positive reals and positive natural numbers by $\mathbb{R}_0$, $\mathbb{R}_+$, and $\mathbb{N}_+$, respectively. In addition, we define $\mathbb{N}_{[i,k]} := \{j \in \mathbb{N} \,|\, i \leq j \leq k\}$. Let $m, n \in \mathbb{N}_+$ and let $z \in \mathbb{R}^{n+m}$. We frequently deal with orthogonal projections of z onto different subspaces. In this context, we define the matrices $P_x := \begin{pmatrix} I_n & 0 \end{pmatrix} \in \mathbb{R}^{n \times (n+m)}$ and $P_u := \begin{pmatrix} 0 & I_m \end{pmatrix} \in \mathbb{R}^{m \times (n+m)}$, where I_j

is the identity matrix in $\mathbb{R}^{j \times j}$. Obviously, $P_x\, z$ results in a vector containing the first n elements of z. For a compact set $\mathcal{Z} \subset \mathbb{R}^{n+m}$, the projection $P_x\, \mathcal{Z}$ is understood as $P_x\, \mathcal{Z} = \{P_x\, z \,|\, z \in \mathcal{Z}\}$. Analogously, scaling of $\mathcal{Z}$ by any factor $\lambda > 0$ is defined as $\lambda\, \mathcal{Z} = \{\lambda\, z \,|\, z \in \mathcal{Z}\}$. By $\mathrm{extr}(\mathcal{Z})$, we denote the set of all extreme points of $\mathcal{Z}$. Moreover, $\mathrm{conv}(\{z_1, \dots, z_l\})$ refers to the convex hull of the points $z_1, \dots, z_l \in \mathbb{R}^{n+m}$. Finally, let $A \in \mathbb{R}^{n \times n}$. Then, $\mu_2(A)$ denotes the so-called logarithmic norm induced by the spectral matrix norm (see [11] for details).

2 Problem statement

Consider the continuous-time linear system

$$\dot{x}(t) = A\, x(t) + B\, u(t), \qquad x(0) = x_0 \tag{1}$$

with (combined) state and input constraints of the form

$$\begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \in \mathcal{Z} \quad \text{for every } t \in \mathbb{R}_0 \tag{2}$$

under piecewise constant control (resp. ZOH)

$$u(t) = u(t_k) \quad \text{for every } t \in [k\, \Delta t, (k+1)\, \Delta t), \tag{3}$$

where $\Delta t \in \mathbb{R}_+$ denotes the sampling time and where $t_k := k\, \Delta t$ for every $k \in \mathbb{N}$. The control task is to steer the system to the origin under the following assumptions.

Assumption 1: *The matrices $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$ are such that the pair (A, B) is stabilizable. The set $\mathcal{Z} \subset \mathbb{R}^{n+m}$ is a convex and compact polytope with the origin in its interior.*

During the controller design (and the controller evaluation), system (1) w.r.t. (2) and (3) is usually replaced by the discrete-time system

$$x(t_{k+1}) = \widehat{A}\, x(t_k) + \widehat{B}\, u(t_k), \qquad x(0) = x_0 \tag{4}$$

with adapted constraints

$$\begin{pmatrix} x(t_k) \\ u(t_k) \end{pmatrix} \in \widehat{\mathcal{Z}} \quad \text{for every} \quad k \in \mathbb{N}, \tag{5}$$

where $\widehat{A} := \exp(A\, \Delta t)$ and $\widehat{B} := \int_0^{\Delta t} \exp(A\, \tau)\, \mathrm{d}\tau\, B$. In practice, the constraints $\widehat{\mathcal{Z}}$ of the discrete-time system are often chosen identical to the ones of the continuous-time system, i.e., $\widehat{\mathcal{Z}} = \mathcal{Z}$. It is, however, well known that the continuous-time system (1) may violate the constraints $\mathcal{Z}$ although the discrete-time system (4) satisfies (5) for $\widehat{\mathcal{Z}} = \mathcal{Z}$. Precisely, let $\mathbf{u}_N = \{u(t_0), \dots, u(t_{N-1})\}$ denote an input sequence of length $N \in \mathbb{N}_+$ and let $\hat{\varphi}(k, x_0, \mathbf{u}_N)$ (resp. $\varphi(t, x_0, \mathbf{u}_N)$) be the solution of (4) (resp. (1) under control policy (3)) at time t_k for $k \in \mathbb{N}_{[0,N]}$ (resp. at time $t \in [0, N\Delta t]$) and initial condition x_0. Then, we may have

$$\begin{pmatrix} \varphi(t, x_0, \mathbf{u}_N) \\ u(t) \end{pmatrix} \notin \mathcal{Z} \quad \text{for some} \quad t \in [0, N\Delta t] \tag{6}$$

even though

$$\begin{pmatrix} \hat{\varphi}(k, x_0, \mathbf{u}_N) \\ u(t_k) \end{pmatrix} \in \mathcal{Z} \quad \text{for every} \quad k \in \mathbb{N}_{[0,N]}.$$

The paper deals with the computation of suitable constraints $\widehat{\mathcal{Z}} \subseteq \mathcal{Z}$ for the discretized system in order to avoid constraint violations of the continuous-time system as in (6).

Definition 1: *We call a set $\widehat{\mathcal{Z}} \subseteq \mathcal{Z}$ a suitable constraint set for the discretized system* (4) *if, for any $x_0 \in \mathbb{R}^n$, any $N \in \mathbb{N}_+$, and any input sequence $\mathbf{u}_N$, constraint satisfaction of the discretized system* (4)*, i.e.,*

$$\begin{pmatrix} \hat{\varphi}(k, x_0, \mathbf{u}_N) \\ u(t_k) \end{pmatrix} \in \widehat{\mathcal{Z}} \quad \text{for every} \quad k \in \mathbb{N}_{[0,N]}, \tag{7}$$

implies constraint satisfaction of the continuous-time system (1) *under control policy* (3)*, i.e.,*

$$\begin{pmatrix} \varphi(t, x_0, \mathbf{u}_N) \\ u(t) \end{pmatrix} \in \mathcal{Z} \quad \text{for every} \quad t \in [0, N\Delta t]. \tag{8}$$

3 Computation of suitable constraint sets

Definition 1 provides an intuitive description of suitable constraints by considering input sequences of (arbitrary) length $N \in \mathbb{N}_+$. Such input sequences are for example investigated in MPC. However, according to Lemma 1, suitable constraint sets can be identified by solely analyzing input sequences of length $N = 1$. As a preparation, note that we write $\varphi(t, x_0, u_0)$ instead of $\varphi(t, x_0, \mathbf{u}_1)$ (resp. $\hat{\varphi}(1, x_0, u_0)$ instead of $\hat{\varphi}(1, x_0, \mathbf{u}_1)$) whenever we consider an input sequence of length $N = 1$, where u_0 is short for $u(t_0)$.

Lemma 1: *Assume the set $\widehat{\mathcal{Z}} \subseteq \mathcal{Z}$ is such that, for every $z_0 \in \widehat{\mathcal{Z}}$, we have*

$$\begin{pmatrix} \varphi(t, x_0, u_0) \\ u_0 \end{pmatrix} \in \mathcal{Z} \quad \text{for every} \quad t \in [0, \Delta t], \tag{9}$$

where $x_0 := P_x z_0$ and $u_0 := P_u z_0$. Then, $\widehat{\mathcal{Z}}$ is a suitable constraint set for (4).

Proof. Consider any $x_0 \in \mathbb{R}^n$, any $N \in \mathbb{N}_+$, and any input sequence $\mathbf{u}_N$ and assume (7) holds. We below show that condition (9) implies (8). Consequently, $\widehat{\mathcal{Z}}$ is suitable constraint set for (4) according to Definition 1.

Let $k^* \in \mathbb{N}_{[0,N-1]}$ be arbitrary but fixed and set $x^* := \hat{\varphi}(k^*, x_0, \mathbf{u}_N)$ and $u^* := u(t_{k^*})$ (where $u(t_{k^*})$ refers to the $(k^* + 1)$-th element of $\mathbf{u}_N$). Let

$$z^* := \begin{pmatrix} x^* \\ u^* \end{pmatrix}$$

and note that $z^* \in \widehat{\mathcal{Z}}$ according to (7). Moreover, we obviously have $\varphi(t, x_0, u_0) = \varphi(t - k^*\Delta t, x^*, u^*)$ and $u(t) = u^*$ for every $t \in [k^*\Delta t, (k^*+1)\Delta t]$ by construction. Thus, we find

$$\begin{pmatrix} \varphi(t, x_0, u_0) \\ u(t) \end{pmatrix} \in \mathcal{Z} \quad \text{for every } t \in [k^*\Delta t, (k^*+1)\Delta t], \tag{10}$$

according to (9). Since $k^* \in \mathbb{N}_{[0,N-1]}$ was arbitrary, (10) implies (8). ■

In the following, we exploit Lemma 1 in order to verify suitable constraint sets.

3.1 Verification

According to Assumption 1, the set $\mathcal{Z}$ is a convex and compact polytope containing the origin as an interior point. Thus, it can written as

$$\mathcal{Z} = \{z \in \mathbb{R}^{n+m} \,|\, H_z\, z \leq \mathbf{1}_h\}, \tag{11}$$

where $H_z \in \mathbb{R}^{h\times(n+m)}$ is a full-rank matrix with $h \geq n+m+1$ and where $\mathbf{1}_h \in \mathbb{R}^h$ is a vector with all entries equal to 1. Clearly, (9) holds if (and only if)

$$\max_{z_0 \in \widehat{\mathcal{Z}}} \max_{j \in \mathbb{N}_{[1,h]}} \max_{t \in [0,\Delta t]} e_j^T (H_x \varphi(t, x_0, u_0) + H_u u_0) \leq 1, \tag{12}$$

where $e_j \in \mathbb{R}^h$ is the j-th Euclidean unit vector and where $H_x := H_z P_x^T$, $H_u := H_z P_u^T$, $x_0 := P_x z_0$, and $u_0 := P_u z_0$.

Remark 1: *Note that condition* (12) *is similar to but different from the counterpart in [2, Thm. 5]. Roughly speaking, the expression in [2, Thm. 5] contains a maximization w.r.t. the state x_0 and a minimization w.r.t. the input u_0, while we consider a maximization w.r.t. the augmented state z_0 in* (12)*. The minimization in [2] finally leads to controlled safe sets (see [2, Def. 2]), which "only" guarantee the existence of a suitable input such that the continuous-time system does not violate the original state and input constraints. In contrast, we are looking for suitable constraint sets according to Definition 1 that guarantee constraint satisfaction of the continuous-time system for all input sequences that respect the adapted constraints $\widehat{\mathcal{Z}}$ of the discretized system. Thus, we have to consider the maximization w.r.t. z_0.*

Taking into account that $\varphi(t, x_0, u_0)$ evaluates to

$$\varphi(t, x_0, u_0) = \exp(A\,t)\, x_0 + \int_0^t \exp(A\,\tau)\,\mathrm{d}\tau\, B\, u_0 \tag{13}$$

for every $t \in [0, \Delta t]$, it is obvious that the expression $e_j^T (H_x \varphi(t, x_0, u_0) + H_u u_0)$ in (12) is in general not concave (nor convex) on $[0, \Delta t]$ for fixed values of x_0, u_0, and e_j. Hence, checking whether the l.h.s. in (12) is smaller than or equal to 1 is a non-convex optimization problem (OP). However, assuming $\widehat{\mathcal{Z}}$ is a convex polytope, we are able to simplify the outer maximization in (12). This is summarized in the following proposition.

Proposition 2: *Assume $\widehat{\mathcal{Z}} \subseteq \mathcal{Z}$ is a convex polytope, which can be written as $\widehat{\mathcal{Z}} = \mathrm{conv}(\{z_1, \dots, z_l\})$, where $z_i \in \mathbb{R}^{n+m}$. Then, $\widehat{\mathcal{Z}}$ is a suitable constraint set for (4) if*

$$\max_{i \in \mathbb{N}_{[1,l]}} \max_{j \in \mathbb{N}_{[1,h]}} \max_{t \in [0,\Delta t]} e_j^T (H_x \varphi(t, v_i, w_i) + H_u w_i) \leq 1, \tag{14}$$

where $v_i := P_x z_i$ and $w_i := P_u z_i$.

Proof. Consider any $z_0 \in \widehat{\mathcal{Z}}$ and note that z_0 can be written as a convex combination of the form $z_0 = \sum_{i=1}^{l} \alpha_i z_i$ for some $\alpha_i \geq 0$ with $\sum_{i=1}^{l} \alpha_i = 1$. Let $x_0 = P_x z_0 = \sum_{i=1}^{l} \alpha_i v_i$ and $u_0 = P_u z_0 = \sum_{i=1}^{l} \alpha_i w_i$. Then, with regard to (13), we easily obtain

$$\begin{aligned} &\max_{t \in [0,\Delta t]} e_j^T (H_x \varphi(t, x_0, u_0) + H_u u_0) \\ &= \sum_{i=1}^{l} \alpha_i \max_{t \in [0,\Delta t]} e_j^T (H_x \varphi(t, v_i, w_i) + H_u w_i) \leq \sum_{i=1}^{l} \alpha_i = 1, \end{aligned}$$

where the inequality holds due to (14). Thus, condition (12) holds and $\widehat{\mathcal{Z}}$ is a suitable constraint set for (4) according to Lemma 1. ∎

Proposition 2 implies that the two outer maximizations in (12) can be reformulated as an enumeration problem. Consequently, checking (12) "only" requires to repeatedly solve the univariate non-convex OP

$$\max_{t \in [0,\Delta t]} e_j^T (H_x \varphi(t, v_i, w_i) + H_u w_i) \tag{15}$$

for all combinations v_i, w_i, and e_j. We discuss the solution of (15) in more detail in Section 3.4. First, however, we address the identification of suitable constraints and comment on the accuracy of the proposed procedure.

3.2 Identification

We will exploit Proposition 2 to identify suitable constraint sets $\widehat{\mathcal{Z}}$. As a preparation, we need a procedure for the computation of appropriate candidate sets. To this end, consider the sequence of sets defined by $\mathcal{Q}_0 := \mathcal{Z}$ and

$$\mathcal{Q}_{k+1} := \left\{ z \in \mathcal{Z} \,\middle|\, \begin{pmatrix} \widehat{A} & \widehat{B} \end{pmatrix} z \in P_x \mathcal{Q}_k \right\} \tag{16}$$

for every $k \in \mathbb{N}$. It is easy to show that $\{\mathcal{Q}_k\}$ results in a sequence of nested sets satisfying

$$\mathcal{Q}_{k+1} \subseteq \mathcal{Q}_k \subseteq \mathcal{Z}$$

for every $k \in \mathbb{N}$ (see [3] for details). We will use $\mathcal{Q}_k$ (for an arbitrary $k \in \mathbb{N}$) as a candidate set for $\widehat{\mathcal{Z}}$. More precisely, we set $\widetilde{\mathcal{Z}} = \mathcal{Q}_k$ and try to identify a suitable constraint set $\widehat{\mathcal{Z}} \subseteq \widetilde{\mathcal{Z}}$ by (non-uniform) scaling of $\widetilde{\mathcal{Z}}$.

Proposition 3: *Let $k \in \mathbb{N}$ and set $\widetilde{\mathcal{Z}} = \mathcal{Q}_k$. Then,* $\mathrm{extr}(\widetilde{\mathcal{Z}}) = \{\tilde{z}_1, \dots, \tilde{z}_l\}$ *is a nonempty finite set. Moreover, let*

$$\varrho_{ij} := \max_{t \in [0,\Delta t]} e_j^T (H_x \varphi(t, \tilde{v}_i, \tilde{w}_i) + H_u \tilde{w}_i) \tag{17}$$

for every $i \in \mathbb{N}_{[1,l]}$ and every $j \in \mathbb{N}_{[1,h]}$, where $\tilde{v}_i := P_x \tilde{z}_i$, and $\tilde{w}_i := P_u \tilde{z}_i$ and set

$$\hat{z}_i := \frac{1}{\max\{1, \varrho_{i1}, \dots, \varrho_{ih}\}} \, \tilde{z}_i. \tag{18}$$

Then $\widehat{\mathcal{Z}} = \mathrm{conv}(\{\hat{z}_1, \dots, \hat{z}_l\})$ is a suitable constraint set for (4)*.*

Proof. We obviously have $0 \in \mathcal{Q}_k$ for every $k \in \mathbb{N}$. Hence $\mathrm{extr}(\widetilde{\mathcal{Z}})$ is nonempty. Moreover, it is easy to show that $\mathcal{Q}_k$ is a polytope for every $k \in \mathbb{N}$, since $\mathcal{Q}_0 = \mathcal{Z}$ is a polytope by assumption. Thus, $\mathrm{extr}(\widetilde{\mathcal{Z}})$ is finite. Consequently, $\widehat{\mathcal{Z}}$ is a convex polytope by construction. Consider any vertex $\hat{z}_i$ of $\widehat{\mathcal{Z}}$ and any $j \in \mathbb{N}_{[1,h]}$ and note that

$$\begin{aligned} &\max_{t \in [0,\Delta t]} e_j^T (H_x \varphi(t, \hat{v}_i, \hat{w}_i) + H_u \hat{w}_i) \\ &= \frac{\max_{t \in [0,\Delta t]} e_j^T (H_x \varphi(t, \tilde{v}_i, \tilde{w}_i) + H_u \tilde{w}_i)}{\max\{1, \varrho_{i1}, \dots, \varrho_{ih}\}} \\ &= \frac{\varrho_{ij}}{\max\{1, \varrho_{i1}, \dots, \varrho_{ih}\}} \leq 1, \end{aligned}$$

where $\hat{v}_i := P_x \hat{z}_i$ and $\hat{w}_i := P_u \hat{z}_i$. Thus, $\widehat{\mathcal{Z}}$ is a suitable constraint set for (4) according to Proposition 2. ■

Proposition 3 suggests to compute suitable constraint sets according to the following simple algorithm.

Algorithm 1: *Computation of suitable constraint sets $\widehat{\mathcal{Z}}$.*

(i) choose any $\Delta t \in \mathbb{R}_+$ and compute $\widehat{A}$ and $\widehat{B}$,

(ii) choose any $k \in \mathbb{N}$, compute $\mathcal{Q}_k$, and set $\widetilde{\mathcal{Z}} = \mathcal{Q}_k$,

(iii) evaluate $\{\tilde{z}_1, \dots, \tilde{z}_l\} = \mathrm{extr}(\widetilde{\mathcal{Z}})$ and compute all ϱ_{ij} according to (17)*,*

(iv) compute $\hat{z}_i$ as in (18) *and set $\widehat{\mathcal{Z}} = \mathrm{conv}(\{\hat{z}_1, \dots, \hat{z}_l\})$.*

3.3 Accuracy

By construction, the adapted constraints $\widehat{\mathcal{Z}}$ computed according to Algorithm 1 are a subset of the original constraints, i.e., $\widehat{\mathcal{Z}} \subseteq \mathcal{Z}$. In most cases, the adapted constraints will in fact be a strict subset of $\mathcal{Z}$, i.e., $\widehat{\mathcal{Z}} \subset \mathcal{Z}$. It is important to note that any choice $\widehat{\mathcal{Z}} \subset \mathcal{Z}$ may entail the exclusion of actually stabilizable states. For example, there may exist states $x_0 \in (P_x \mathcal{Z}) \setminus (P_x \widehat{\mathcal{Z}})$ which are stabilizable for the continuous-time system w.r.t. $\mathcal{Z}$ but unstabilizable for the discretized system w.r.t. $\widehat{\mathcal{Z}}$ since $x_0 \notin P_x \widehat{\mathcal{Z}}$. While this effect cannot be completely avoided, we show in the following that a suitable choice of the sampling time Δt allows to bound the defect of $\widehat{\mathcal{Z}}$.

At first, it is crucial to note that the computation of a candidate set $\widetilde{\mathcal{Z}}$ according to step (ii) in Algorithm 1 does not exclude any stabilizable states. In fact, if $x_0 \notin P_x \mathcal{Q}_k$ for some $k \in \mathbb{N}$ then x_0 cannot be stabilizable for the continuous-time system since any trajectory emanating from x_0 is guaranteed to violate the constraints $\mathcal{Z}$ for some points in time in the interval $[0, k\,\Delta t]$ (cf. [3]). Consequently, the only source of conservatism is the (non-uniform) scaling in (18) based on the factors ϱ_{ij} from (17). However, the defect resulting from scaling can be easily bounded. Assume for the moment that

$$\max_{i \in \mathbb{N}_{[1,l]}} \max_{j \in \mathbb{N}_{[1,h]}} \varrho_{ij} \leq 1 + \epsilon \tag{19}$$

for some $\epsilon \geq 0$. Then, we obviously have

$$\frac{1}{1+\epsilon} \widetilde{\mathcal{Z}} \subseteq \widehat{\mathcal{Z}} \subseteq \widetilde{\mathcal{Z}}. \tag{20}$$

Thus, ϵ provides a measure for the defect of $\widehat{\mathcal{Z}}$. Obviously, evaluating the l.h.s. of (19) allows to *a posteriori* quantify the defect ϵ for a given set $\widehat{\mathcal{Z}}$. However, it is also possible to *a priori* choose Δt and k such that the adapted constraint set $\widehat{\mathcal{Z}}$ resulting from Algorithm 1 is guaranteed to satisfy (20) for a given $\epsilon > 0$. This is summarized in Corollary 5 further below. As a preparation, we prove the following proposition.

Proposition 4: *Let $\epsilon \in \mathbb{R}_+$ and let $\tilde{z}_1, \dots, \tilde{z}_l \in \mathbb{R}^{n+m}$ with $l \in \mathbb{N}_+$. Let $c_1, c_2, c_3 \in \mathbb{R}_+$ be such that $c_1 \geq \max_{j \in \mathbb{N}_{[1,h]}} \|e_j^T H_x\|_2$, $c_2 \geq \sqrt{\|A\|_2^2 + \|B\|_2^2}$, and $c_3 \geq \max_{i \in \mathbb{N}_{[1,l]}} \|\tilde{z}_i\|_2$. Finally, let $\Delta t \in \mathbb{R}_+$ be such that*

$$\Delta t \leq \begin{cases} \frac{1}{\mu_2(A)} \ln\left(\frac{\epsilon\,\mu_2(A)}{c_1\,c_2\,c_3} + 1\right) & \text{if } \mu_2(A) > 0, \\ \frac{\epsilon}{c_1\,c_2\,c_3} & \text{otherwise.} \end{cases} \tag{21}$$

Then (19) *holds, where ϱ_{ij} is defined as in* (17).

Proof. The proof is inspired by [10, Lem. 11]. Consider any $i \in \mathbb{N}_{[1,l]}$ and $j \in \mathbb{N}_{[1,h]}$ and note (17) can be written as

$$\begin{aligned} &\max_{t \in [0,\Delta t]} e_j^T (H_x(\varphi(t,\tilde{x}_i,\tilde{u}_i) - \tilde{x}_i + \tilde{x}_i) + H_u \tilde{u}_i) \\ &\quad = \max_{t \in [0,\Delta t]} e_j^T H_z \tilde{z}_i + e_j^T H_x(\varphi(t,\tilde{x}_i,\tilde{u}_i) - \tilde{x}_i). \end{aligned} \tag{22}$$

Obviously, since $\tilde{z}_i \in \widetilde{\mathcal{Z}} \subseteq \mathcal{Z}$, we obtain the upper bound $e_j^T H_z \tilde{z}_i \leq 1$ for the first term in (22). It remains to show

$$\max_{t \in [0,\Delta t]} e_j^T H_x(\varphi(t,\tilde{x}_i,\tilde{u}_i) - \tilde{x}_i) \leq \epsilon. \tag{23}$$

Clearly, the l.h.s. in (23) is dominated by

$$\begin{aligned} &\max_{t \in [0,\Delta t]} |e_j^T H_x(\varphi(t,\tilde{x}_i,\tilde{u}_i) - \tilde{x}_i)| \\ &\leq \max_{t \in [0,\Delta t]} \|e_j^T H_x\|_2 \, \|\varphi(t,\tilde{x}_i,\tilde{u}_i) - \tilde{x}_i\|_2, \\ &\leq c_1 \max_{t \in [0,\Delta t]} \|\varphi(t,\tilde{x}_i,\tilde{u}_i) - \tilde{x}_i\|_2, \end{aligned} \tag{24}$$

where the last inequality holds by definition of c_1. In order to overestimate the maximization in (24), we consider the time-derivative

$$\dot{\varphi}(t, \tilde{x}_i, \tilde{u}_i) = A \exp(A\, t)\, \tilde{x}_i + \exp(A\, t)\, B\, \tilde{u}_i \tag{25}$$

and we note that the arc length $s(t, \tilde{x}_i, \tilde{u}_i)$ of the trajectory $\varphi(t, \tilde{x}_i, \tilde{u}_i)$ at time $t \in [0, \Delta t]$ emanating from $\tilde{x}_i$ can be written as

$$s(t, \tilde{x}_i, \tilde{u}_i) = \int_0^t \|\dot{\varphi}(\tau, \tilde{x}_i, \tilde{u}_i)\|_2 \, \mathrm{d}\tau. \tag{26}$$

By construction, we obtain

$$\|\varphi(t, \tilde{x}_i, \tilde{u}_i) - \tilde{x}_i\|_2 \leq s(t, \tilde{x}_i, \tilde{u}_i) \tag{27}$$

for very $t \in [0, \Delta t]$. Now, the arc length can be overestimated as follows. By substituting (25) in (26), we obtain

$$\begin{aligned}
s(t, \tilde{x}_i, \tilde{u}_i) &= \int_0^t \left\| \begin{pmatrix} A \exp(A\, \tau) & \exp(A\, \tau)\, B \end{pmatrix} \tilde{z}_i \right\|_2 \mathrm{d}\tau \\
&\leq c_3 \int_0^t \left\| \begin{pmatrix} A \exp(A\, \tau) & \exp(A\, \tau)\, B \end{pmatrix} \right\|_2 \mathrm{d}\tau \qquad (28) \\
&\leq c_3 \int_0^t \sqrt{\|A \exp(A\, \tau)\|_2^2 + \|\exp(A\, \tau)\, B\|_2^2} \, \mathrm{d}\tau \\
&\leq c_3 \sqrt{\|A\|_2^2 + \|B\|_2^2} \int_0^t \|\exp(A\, \tau)\|_2 \, \mathrm{d}\tau \\
&\leq c_2\, c_3 \int_0^t \exp(\mu_2(A)\, \tau) \, \mathrm{d}\tau, \qquad (29)
\end{aligned}$$

where the second relation holds since $\|\tilde{z}_i\|_2 \leq c_3$. The third relation holds according to [4, Thm. 1][1]. Finally, the two last relations hold according to [11, Prop. 2.1] and by definition of c_2, respectively. Depending on the value of $\mu_2(A)$, the integral in (29) evaluates to[2]

$$\int_0^t \exp(\mu_2(A)\, \tau) \, \mathrm{d}\tau = \begin{cases} \frac{\exp(\mu_2(A)\, t) - 1}{\mu_2(A)} & \text{if } \mu_2(A) \neq 0, \\ t & \text{otherwise.} \end{cases} \tag{30}$$

Taking the maximum on the domain $[0, \Delta t]$ obviously yields

$$\max_{t \in [0, \Delta t]} \frac{\exp(\mu_2(A)\, t) - 1}{\mu_2(A)} = \frac{\exp(\mu_2(A)\, \Delta t) - 1}{\mu_2(A)}, \tag{31}$$

if $\mu_2(A) \neq 0$, and $\max_{t \in [0, \Delta t]} t = \Delta t$ otherwise. Now, since (31) is smaller than Δt whenever $\mu_2(A) < 0$, we may use the following overestimation:

---

[1] Note that the Schatten ∞-norm investigated in [4] is identical to the spectral norm $\|\cdot\|_2$ of a matrix. Further note the 1×2 block matrix in (28) can easily be extended to a 2×2 block matrix, as required for [4, Thm. 1], by inserting zero blocks. This extension does obviously not affect the validity of the derived relations since $\|Cz\|_2 = \|\left(\begin{smallmatrix} C \\ 0 \end{smallmatrix}\right) z\|_2$.

[2] It is important to note that $\mu_2(A)$ may be zero even if $A \neq 0$. Moreover, we have $\mu_2(A) < 0$ for some matrices A. Thus, the so-called logarithmic “norm” does not match the general requirements for a proper norm.

$$\max_{t\in[0,\Delta t]} \int_0^t \exp(\mu_2(A)\,\tau)\,\mathrm{d}\tau$$
$$\leq \begin{cases} \frac{\exp(\mu_2(A)\,\Delta t)-1}{\mu_2(A)} & \text{if } \mu_2(A) > 0, \\ \Delta t & \text{otherwise.} \end{cases} \tag{32}$$

Evaluating the l.h.s. of (23) finally yields

$$\max_{t\in[0,\Delta t]} e_j^T H_x(\varphi(t,\tilde{x}_i,\tilde{u}_i) - \tilde{x}_i)$$
$$\leq c_1 \max_{t\in[0,\Delta t]} \|\varphi(t,\tilde{x}_i,u_0) - \tilde{x}_i\|_2$$
$$\leq c_1\, c_2\, c_3 \begin{cases} \frac{\exp(\mu_2(A)\,\Delta t)-1}{\mu_2(A)} & \text{if } \mu_2(A) > 0, \\ \Delta t & \text{otherwise.} \end{cases} \tag{33}$$

according to (24), (27), (29), and (32). Substituting the upper bound for Δt from (21) in (33), easily proves (23). ■

Corollary 5: *Let $\epsilon \in \mathbb{R}_+$, $k \in \mathbb{N}$, and let c_1 and c_2 be defined as in Proposition 4. Let $c_3, \Delta t \in \mathbb{R}_+$ be such that $c_3 \geq \max_{z\in\mathcal{Z}} \|z\|_2$ and such that* (21) *holds. Then, $\widehat{\mathcal{Z}}$ computed according to Algorithm 1 satisfies* (20).

Proof. We have $\widetilde{\mathcal{Z}} = \mathcal{Q}_k \subseteq \mathcal{Z}$ by construction for any $k \in \mathbb{N}$. Since $\widetilde{\mathcal{Z}}$ is a nonempty polytope (see proof of Prop. 3), $\{\tilde{z}_1,\dots,\tilde{z}_l\} = \mathrm{extr}(\widetilde{\mathcal{Z}})$ is a finite set with $l \in \mathbb{N}_+$. Moreover, we have $\max_{i\in\mathbb{N}_{[1,l]}} \|\tilde{z}_i\|_2 \leq c_3$. Thus, the claim in Corollary 5 directly follows from Proposition 4. ■

3.4 Implementation and Complexity

The computation of suitable constraint sets using Algorithm 1 requires to solve a finite number of univariate non-convex OPs of the form (17) (resp. (15)). Following the argumentation in [2], although the problem is non-convex, it can be solved reliable since it is the search of the maximum of a scalar function on a scalar compact domain. One way to concretely solve (17) is to use interval arithmetics (see [9]) and bisection to identify intervals $[\underline{t},\overline{t}] \subseteq [0,\Delta t]$ on which the objective function is either non-decreasing, non-increasing, convex, or concave. Obviously, on these intervals, the local maximum can be easily computed and the global maximum results from enumeration. Note that interval inclusions for the matrix $\exp(A\,t)$ on intervals $t \in [\underline{t},\overline{t}]$ can be computed according to [**?**]. Further details on the solution of (17) are beyond the scope of the paper. We stress, however, that we successfully applied the described procedure to compute $\widehat{\mathcal{Z}}$ for the numerical example in Section 4.

With regard to (17), the number of OPs to be solved in step (iv) of Algorithm 1 depends on the number of vertices l of the candidate set $\widetilde{\mathcal{Z}}$ and the number of hyperplanes h defining the original constraint set $\mathcal{Z}$. While the number h is inherently fixed, the number l (strongly) depends on the choice of k in step (ii) of Algorithm 1. In fact, it is well-known that the number of vertices of the set $\mathcal{Q}_k$ grows with k (see, e.g., [6, Rem. 4.8]). Fortunately, Algorithm 1 provides a suitable constraint set for every choice of k including $k = 0$ (which implies $\mathcal{Q}_0 = \mathcal{Z}$). Moreover, according to Corollary 5, a suitable choice of the sampling time Δt allows us to meet any desired accuracy $\epsilon > 0$ independent of the choice of k. Nevertheless, a larger k allows to eliminate unstabilizable states and unsuitable inputs before applying the scaling (18). Thus, choosing a larger k may permit to achieve a particular accuracy without having to use undesired small sampling times Δt. In summary, the choice of k allows to control the trade-off between the numerical effort to compute the adapted constraints and their accuracy.

4 Numerical Example

We consider the example discussed in [12] with

$$A = \begin{pmatrix} -0.7 & 0.1 \\ 2.0 & -0.1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2.0 \\ 1.0 \end{pmatrix}$$

and the constraints

$$-2 \leq x_1(t) \leq 2, \ -2 \leq x_2(t) \leq 2, \ \text{and} \ -1 \leq u(t) \leq 1 \tag{34}$$

for every $t \in \mathbb{R}_0$. Clearly, the constraints (34) can be rewritten in the form (2) with $\mathcal{Z}$ as in (11) and

$$H_z = \begin{pmatrix} 0.5 & -0.5 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.5 & -0.5 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & -1.0 \end{pmatrix}^T \in \mathbb{R}^{6\times 3}. \tag{35}$$

We apply Algorithm 1 to compute suitable constraint sets $\widehat{\mathcal{Z}}$ for all combinations of Δt and k satisfying $\Delta t \in \{0.2, 0.5, 1.0\}$ and $k \in \{0, 2\}$. To make our results more transparent, we provide details on the set $\widehat{\mathcal{Z}}$ for the choice $\Delta t = 0.5$ and $k = 2$. In this case, Algorithm 1 yields $\widehat{\mathcal{Z}} = \{z \in \mathbb{R}^3 \,|\, \widehat{H}_z z \leq \mathbf{1}_{18}\}$ with

$$\widehat{H}_z = \begin{pmatrix} 0.4796 & 0.3345 & 0.5528 \\ 0.0000 & -0.5000 & 0.0000 \\ 0.0000 & 0.0000 & 1.0000 \\ -0.0240 & -0.1282 & 0.7841 \\ -0.0240 & -0.5165 & 0.0075 \\ 0.0240 & 0.5165 & -0.0075 \\ 0.0000 & 0.5000 & 0.0000 \\ -0.3621 & -0.0207 & -0.4309 \\ -0.5000 & 0.0000 & 0.0000 \\ -0.4796 & -0.3345 & -0.5528 \\ -0.3457 & -0.5059 & -0.3544 \\ -0.4143 & -0.4864 & -0.4661 \\ 0.3621 & 0.0207 & 0.4309 \\ 0.0240 & 0.1282 & -0.7841 \\ 0.3457 & 0.5059 & 0.3544 \\ 0.0000 & 0.0000 & -1.0000 \\ 0.4143 & 0.4864 & 0.4661 \\ 0.5000 & 0.0000 & 0.0000 \end{pmatrix} \in \mathbb{R}^{18\times 3}. \tag{36}$$

Figure 1 illustrates $\widehat{\mathcal{Z}}$ with $\widehat{H}_z$ from (36). A projection of $\widehat{\mathcal{Z}}$ onto the state space is depicted in Figure 2 further below. For every combination of Δt and k, we a posteriori calculate the defect ϵ of $\widehat{\mathcal{Z}}$ according to (19) as

$$\epsilon = \max_{i \in \mathbb{N}_{[1,l]}} \max_{j \in \mathbb{N}_{[1,h]}} \varrho_{ij} - 1,$$

where the factors ϱ_{ij} result from (17). Numerical values for ϵ regarding all six combinations of Δt and k are listed in Table 1. Obviously, the smallest defect $\epsilon = 0.0111$ results for the choice $\Delta t = 0.2$ and $k = 2$. Moreover, it is interesting to note that the defect $\epsilon = 0.2714$ for the combination $k = 2$ and $\Delta t = 1.0$ is smaller than $\epsilon = 0.4910$ for the choice $k = 0$ and $\Delta t = 0.2$. Thus, eliminating unstabilizable states and unsuitable inputs (by computing $\widetilde{\mathcal{Z}} = \mathcal{Q}_k$ for $k > 0$) before applying the non-uniform scaling (18) allows to choose larger sampling times without increasing the defect of $\widehat{\mathcal{Z}}$.

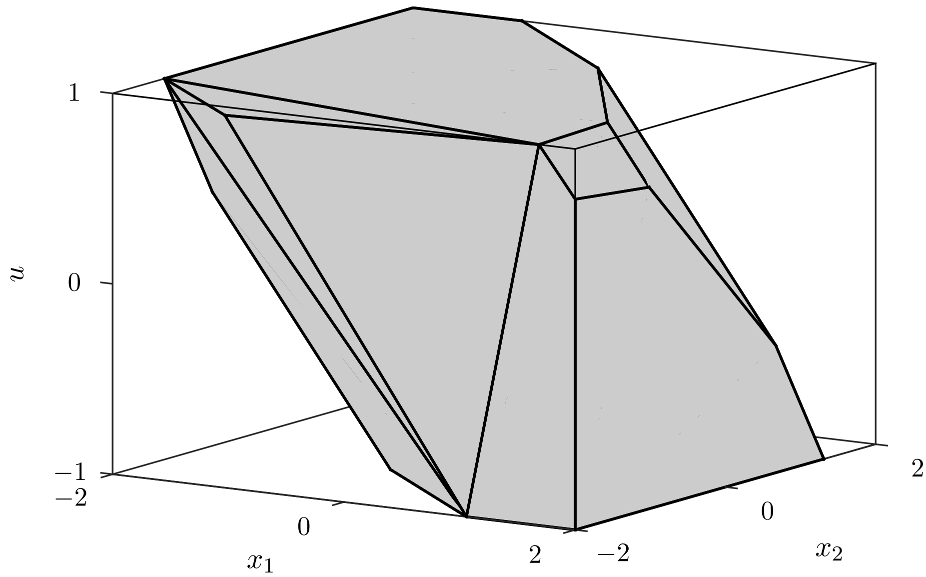

Figure 1: Illustration of the suitable constraint set $\widehat{\mathcal{Z}}$ computed for the example in Section 4 and the choice $\Delta t = 0.5$ and $k = 2$. The axis-aligned bounding box $[x_1] \times [x_2] \times [u] = [-2, 2] \times [-2, 2] \times [-1, 1]$ refers to the original constraint set $\mathcal{Z}$ of the continuous-time system.

Now, assume we want to a priori choose Δt and k such that $\widehat{\mathcal{Z}}$ resulting from Algorithm 1 satisfies (20) with $\epsilon = 0.0111$ as above. According to Corollary 5, this is guaranteed for an arbitrary choice of $k \in \mathbb{N}$ if Δt satisfies (21) with c_1 and c_2 as in Proposition 4 and c_3 as in Corollary 5. To evaluate the r.h.s. in (21), first note that the logarithmic norm of A reads $\mu_2(A) = 0.6920 > 0$. Moreover, the choices

$$\begin{aligned} c_1 &= \max_{j \in \mathbb{N}_{[1,h]}} \|e_j^T H_x\|_2 &&= 0.5000, \\ c_2 &= \sqrt{\|A\|_2^2 + \|B\|_2^2} &&= 3.0832, \quad \text{and} \\ c_3 &= \max_{z \in \mathcal{Z}} \|z\|_2 &&= 3.0000 \end{aligned}$$

are as required in Corollary 5. Consequently, any sampling time $\Delta t \in \mathbb{R}_+$ such that

$$\Delta t \leq \frac{1}{\mu_2(A)} \ln \left(\frac{\epsilon\, \mu_2(A)}{c_1\, c_2\, c_3} + 1 \right) = 0.0024$$

results in a suitable constraint set $\widehat{\mathcal{Z}}$ that satisfies (20) with $\epsilon = 0.0111$. Recall that we already achieved the desired accuracy for the choice $\Delta t = 0.2 \gg 0.0024$ (and $k = 2$). Obviously, a priori bounds for Δt provided by Corollary 5 are valid but conservative.

Table 1: Defects ϵ of the suitable constraint sets $\widehat{\mathcal{Z}}$ (left) and number of vertices l of the candidate sets $\widetilde{\mathcal{Z}}$ (right) computed for the example in Section 4 and different choices of Δt and k.

defect ϵ of $\widehat{\mathcal{Z}}$

k	Δt = 1.0	Δt = 0.5	Δt = 0.2
0	2.6637	1.2676	0.4910
2	0.2714	0.0690	0.0111

vertices l of $\widetilde{\mathcal{Z}}$

k	Δt = 1.0	Δt = 0.5	Δt = 0.2
0	8	8	8
2	16	20	20

In the following, we briefly comment on the numerical complexity associated with the computation of $\widehat{\mathcal{Z}}$ for the present example. As discussed in Section 3.4, the computational effort of Algorithm 1 is mainly determined by the number $h \cdot l$ of OPs to be solved in step (iii). According to (35), the original constraint set $\mathcal{Z}$ is defined by $h = 6$ hyperplanes. The number of vertices l of the candidate set $\widetilde{\mathcal{Z}} = \mathcal{Q}_k$ depends on the choice of k and Δt. Clearly, since $\mathcal{Z}$ is a hyperrectangle in $\mathbb{R}^3$, we obtain $l = 8$ for $k = 0$ and any $\Delta t > 0$. For $k = 2$, the number of vertices l varies with Δt as summarized in Table 1. Obviously, the computation of the set $\widehat{\mathcal{Z}}$ for the combination $k = 2$ and $\Delta t = 0.5$ requires to solve $h \cdot l = 6 \cdot 20 = 120$ OPs of the form (17). Rigorously solving these 120 non-convex OP using the procedure sketched in Section 3.4 (implemented in Matlab R2014b) required 1.6103 s on an Intel Core i5 processor running at 1.9 GHz).

Figure 2 allows to compare the results of the proposed method for the computation of suitable constraints sets with the procedure described in [12]. Obviously, the set $P_x \widehat{\mathcal{Z}}$ is significantly larger than the feasible domain of the MPC scheme designed in [12] (cf. the light gray polytope with the polytope marked by the dash-dotted boundary lines in Fig. 2). This observation is not crucial, however, since the constraint set $\widehat{\mathcal{Z}}$ may contain unstabilizable states. In fact, according to Definition 1, the constraints $\widehat{\mathcal{Z}}$ only guarantee that, if there exists a feasible trajectory for the discretized system (4) w.r.t. (5), then the corresponding trajectory of the continuous-time system satisfies the original constraints $\mathcal{Z}$.

To analyze the set of stabilizable states of system (4) w.r.t. (5), we integrate $\widehat{\mathcal{Z}}$ in a MPC scheme with guaranteed stability and compute the domain of attraction of the closed-loop system. Note, however, that our approach is not restricted to MPC. In fact, the set $\widehat{\mathcal{Z}}$ can be easily integrated in any control scheme that takes constraints explicitly into account. As summarized in [8], stabilizing MPC (for linear systems) is based on the (convex) OP

$$\mathbf{u}_N^* = \min_{\mathbf{u}_N} \|x(t_N)\|_P^2 + \sum_{j=0}^{N-1} \|x(t_j)\|_Q^2 + \|u(t_j)\|_R^2 \tag{37}$$

$$\begin{aligned}
\text{s.t.} \quad & x(0) = x_0, \\
& x(t_{j+1}) = \widehat{A}x(t_j) + \widehat{B}u(t_j), && \text{for } j = 0, \dots, N-1, \\
& \begin{pmatrix} x(t_j) \\ u(t_j) \end{pmatrix} \in \widehat{\mathcal{Z}}, && \text{for } j = 0, \dots, N-1, \\
& x(t_N) \in \mathcal{T},
\end{aligned}$$

where $\widehat{\mathcal{Z}}$ is a suitable constraint set and where $\mathcal{T}$ is an appropriate terminal set (see Eq. (38) further below).

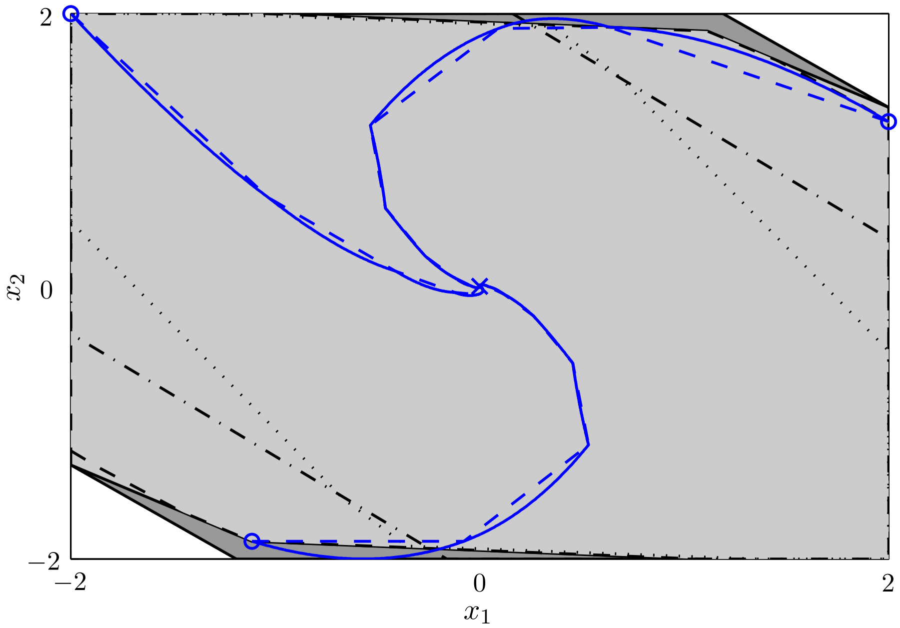

Figure 2: Illustration of the projected suitable constraint set and the predictive control scheme designed for the example in Sect. 4. The dark gray and the light gray polytopes refer to $P_x\widetilde{\mathcal{Z}}$ and $P_x\widehat{\mathcal{Z}}$, respectively, where $\widetilde{\mathcal{Z}}$ and $\widehat{\mathcal{Z}}$ were computed using Alg. 1 for the choice $\Delta t = 0.5$ and $k = 2$. The polytopes with the (black) dashed and dash-dotted boundary lines mark the feasible set (i.e., the domain of attraction) of the MPC schemes designed here and in [12], respectively. The polytope with the (black) dotted boundary line refers to the terminal set $\mathcal{T}$ characterized in (38). The blue lines mark trajectories of the continuous-time system (solid) and the discretized system (dashed) under the predictive control scheme (37) for three different initial conditions x_0 (located at the vertices of the feasible set).

The weighting matrices Q and R and the sampling time Δt are chosen as in [12], i.e., $Q = I_2$, $R = 2$, and $\Delta t = 0.5$. Hence, the system matrices of the discretized system evaluate to

$$\widehat{A} = \begin{pmatrix} 0.7243 & 0.0414 \\ 0.8287 & 0.9729 \end{pmatrix} \quad \text{and} \quad \widehat{B} = \begin{pmatrix} 0.8617 \\ 0.9321 \end{pmatrix}.$$

We consider the constraint set $\widehat{\mathcal{Z}}$ with $\widehat{H}_z$ as in (36) (i.e., the set $\widehat{\mathcal{Z}}$ that results from Algorithm 1 for $\Delta t = 0.5$ and $k = 2$). The terminal weighting P is chosen as the solution of the discrete-time matrix Riccati equation (DARE) and thus accounts for the infinite horizon cost of the discretized system under the stabilizing control law $u(t_k) = K\,x(t_k)$ with $K = -(R + \widehat{B}^T P \widehat{B})^{-1} \widehat{B} P \widehat{A}$. Numerical values for P and K are

$$P = \begin{pmatrix} 2.0265 & 0.6980 \\ 0.6980 & 1.9557 \end{pmatrix} \text{ and } K = \begin{pmatrix} -0.5921 & -0.3886 \end{pmatrix}.$$

In order to guaranty asymptotic stability of the closed-looop system controlled by the MPC scheme (37) (see [8] for details), we consider the terminal set

$$\mathcal{T} = \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{pmatrix} I_n \\ K \end{pmatrix} (\widehat{A} + \widehat{B}K)^k \, x \in \widehat{\mathcal{Z}},\, \forall k \in \mathbb{N} \right\} \tag{38}$$

as introduced in [5]. Note that $\mathcal{T}$ as in (38) can always be described by only considering a finite number of steps $k \in \mathbb{N}$. In fact, for this example, it is sufficient to solely consider $k = 0$. The resulting terminal set $\mathcal{T}$ is illustrated in Figure 2. Finally, we choose the (relatively short) prediction horizon $N = 2$.

In Figure 2, the feasible set of the quadratic program (37) (i.e., the set of all feasible initial conditions $x_0 \in \mathbb{R}^n$) is illustrated. Clearly, the feasible set of the MPC scheme presented here is significantly larger than the feasible set of the predictive controller in [12] even though the prediction horizon in [12] is longer (in fact, $N = 8$ in [12]). To see this, compare the sizes of the polytopes marked by the dashed boundary lines and the dash-dotted boundary lines in Figure 2, respectively. Since both MPC schemes guarantee closed-loop stability, the feasible sets are equal to the domains of attraction of the controlled systems in both cases. Obviously, for this example, the adapted constraints computed with the presented method result in a larger domain of attraction than the constraints proposed in [12]. A reason for this observation may be that the polytopic overapproximations exploited in [12] are more conservative (at least for small choices of the parameter ν (see [12, Lem. 2 and Sect. V])) than the (non-uniform) scaling introduced here (see Eq. (18)).

Finally, Figure 2 illustrates trajectories of the discretized system under the predictive control scheme (37) for three different initial conditions x_0. Analyzing the corresponding trajectories of the continuous-time system proves that the adapted constraints $\widehat{\mathcal{Z}}$ are suitable. In fact, the trajectories of the continuous-time system do not violate the original constraints (34).

5 Conclusion and outlook

We presented a new method to guarantee constraint satisfaction for linear continuous-time systems under piecewise constant control. The approach builds on the systematic adaptation of the state and input constraints such that constraint satisfaction of the discretized system w.r.t. the adapted constraints implies constraint satisfaction of the continuous-time system w.r.t. the original constraints. Roughly speaking, the computation of adapted constraints for the discrete-time system (according to Alg. 1) is based on the idea to first eliminate unstabilizable states (and unsuitable inputs) and then scale the resulting set to obtain a suitable constraint set (according to Def. 1). Compared to existing approaches (see [1, 2, 7, 12]), the new method is less conservative, not restricted to any specific control scheme (e.g., MPC), and computationally less demanding.

Future work has to address efficient numerical procedures for the solution of the recurrent non-convex optimization problem (17) (resp. (15)). Moreover, extensions of the proposed method to time-variant linear systems and the inclusion of disturbances are of interest.

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