Solutions of Selected Problems

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Chapter I

1.9 Consider the potential equation in the disk $\Omega := \{(x, y) \in \mathbb{R}^2;\ x^2+y^2 < 1\}$, with the boundary condition

$$\frac{\partial}{\partial r} u(x) = g(x) \quad \text{for } x \in \partial\Omega$$

on the derivative in the normal direction. Find the solution when g is given by the Fourier series

$$g(\cos\phi, \sin\phi) = \sum_{k=1}^{\infty} (a_k \cos k\phi + b_k \sin k\phi)$$

without a constant term. (The reason for the lack of a constant term will be explained in Ch. II, §3.)

Solution. Consider the function

$$u(r, \phi) := \sum_{k=1}^{\infty} \frac{r^k}{k} (a_k \cos k\phi + b_k \sin k\phi). \tag{1.20}$$

Since the partial derivatives $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \phi}$ refer to orthogonal directions (on the unit circle), we obtain $\frac{\partial}{\partial r} u$ by evaluating the derivative of (1.20). The values for $r = 1$ show that we have a solution. Note that the solution is unique only up to a constant. □

1.12 Suppose u is a solution of the wave equation, and that at time $t = 0$, u is zero outside of a bounded set. Show that the energy

$$\int_{\mathbb{R}^d} [u_t^2 + c^2 (\operatorname{grad} u)^2]\, dx \tag{1.19}$$

is constant.

Hint: Write the wave equation in the symmetric form

$$u_t = c \operatorname{div} v,$$
$$v_t = c \operatorname{grad} u,$$

and represent the time derivative of the integrand in (1.19) as the divergence of an appropriate expression.

Solution. We take the derivative of the integrand and use the differential equations

$$\begin{aligned}
&\frac{\partial}{\partial t}\int_{\mathbb{R}^d}[u_t^2 + c^2(\operatorname{grad} u)^2]\,dx \\
&= \frac{\partial}{\partial t}\int_{\mathbb{R}^d} c^2[(\operatorname{div} v)^2 + (\operatorname{grad} u)^2]\,dx \\
&= c^2\int_{\mathbb{R}^d}[2\operatorname{div} v\operatorname{div}\frac{\partial}{\partial t}v + 2\operatorname{grad} u\operatorname{grad}\frac{\partial}{\partial t}u\,dx \\
&= 2c^3\int_{\mathbb{R}^d}[\operatorname{div} v\operatorname{div}\operatorname{grad} u + \operatorname{grad} u\operatorname{grad}\operatorname{div} v]\,dx \\
&= 2c^3\int_{\mathbb{R}^d}\operatorname{div}[\operatorname{div} v\operatorname{grad} u]\,dx.
\end{aligned}$$

The integrand vanishes outside the interior of a bounded set Ω. By Gauss' integral theorem the integral above equals

$$2c^3\int_{\partial\Omega}\operatorname{div} v\operatorname{grad} u\cdot n ds = 0. \qquad \square$$

Chapter II

1.10 Let Ω be a bounded domain. With the help of Friedrichs' inequality, show that the constant function $u = 1$ is not contained in $H_0^1(\Omega)$, and thus $H_0^1(\Omega)$ is a proper subspace of $H^1(\Omega)$.

Solution. If the function $u = 1$ would belong to H_0^1, then Friedrichs' inequality would imply $\|u\|_0 \le c|u|_1 = 0$. This contradicts $\|u\|_0 = \mu(\Omega)^{1/2} > 0$. $\square$

1.12 A variant of Friedrichs' inequality. Let Ω be a domain which satisfies the hypothesis of Theorem 1.9. Then there is a constant $c = c(\Omega)$ such that

$$\|v\|_0 \le c\big(|\bar v| + |v|_1\big) \quad \text{for all } v\in H^1(\Omega) \tag{1.11}$$
$$\text{with} \quad \bar v = \frac{1}{\mu(\Omega)}\int_\Omega v(x)dx.$$

Hint: This variant of Friedrichs' inequality can be established using the technique from the proof of the inequality 1.5 only under restrictive conditions on the domain. Use the compactness of $H^1(\Omega) \hookrightarrow L_2(\Omega)$ in the same way as in the proof of Lemma 6.2 below.

Solution. Suppose that (1.11) does not hold. Then there is a sequence $\{v_n\}$ such that

$$\|v_n\| = 1 \quad \text{and} \quad |\bar{v}_n| + |v_n|_1 \le n \quad \text{for all } n = 1, 2, \ldots .$$

Since $H^1(\Omega) \hookrightarrow L_2(\Omega)$ is compact, a subsequence converges in $L_2(\Omega)$. After going to a subsequence if necessary, we assume that the sequence itself converges. It is a Cauchy sequence in $L_2(\Omega)$. The triangle inequality yields $|v_n - v_m|_1 \le |v_n|_1 + |v_m|_1$, and $\{v_n\}$ is a Cauchy sequence in $H^1(\Omega)$.

Let $u = \lim_{n\to\infty} v_n$. From $|u|_1 = \lim_{n\to\infty} |v_n|_1 = 0$ it follows that u is a constant function, and from $\bar{u} = 0$ we conclude that $u = 0$. This contradicts $\|u\|_0 = \lim_{n\to\infty} \|v_n\|_0 = 1$. □

1.14 Exhibit a function in $C[0,1]$ which is not contained in $H^1[0,1]$. – To illustrate that $H_0^0(\Omega) = H^0(\Omega)$, exhibit a sequence in $C_0^\infty(0,1)$ which converges to the constant function $v = 1$ in the $L_2[0,1]$ sense.

Solution. Let $0 < \alpha < 1/2$. The function $v := x^\alpha$ is continuous on $[0,1]$, but $v' = \alpha x^{\alpha-1}$ is not square integrable, i.e., $v' \notin L_2[0,1]$. Hence, $v \in C[0,1]$ and $v \notin H^1[0,1]$.

Consider the sequence

$$v_n := 1 + e^{-n} - e^{-nx} - e^{-n(1-x)}, \quad n = 1, 2, 3, \ldots$$

Note that the deviation of v_n from 1 is very small for $e^{-\sqrt{n}} < x < 1 - e^{-\sqrt{n}}$, and that there is the obvious uniform bound $|v_n(x)| \le 2$ in $[0,1]$. Therefore, $\{v_n\}$ provides a sequence as requested. □

1.15 Let ℓ_p denote the space of infinite sequences $(x_1, x_2, \ldots)$ satisfying the condition $\sum_k |x_k|^p < \infty$. It is a Banach space with the norm

$$\|x\|_p := \|x\|_{\ell_p} := \Big(\sum_k |x_k|^p\Big)^{1/p}, \quad 1 \le p < \infty.$$

Since $\|\cdot\|_2 \le \|\cdot\|_1$, the imbedding $\ell_1 \hookrightarrow \ell_2$ is continuous. Is it also compact?

Solution. For completeness we note that $\sum_i |x_i|^2 \le \big(\sum_i |x_i|\big)^2$, and $\|x\|_2 \le \|x\|_1$ is indeed true.

Next consider the sequence $\{x^j\}_{j=1}^\infty$, where the $j-th$ component of x^j equals 1 and all other components vanish. Obviously, the sequence belongs to the unit ball in ℓ_1, but there is no subsequence that converges in ℓ_2. The imbedding is not compact. □

1.16 Consider
(a) the Fourier series $\sum_{k=-\infty}^{+\infty} c_k e^{ikx}$ on $[0, 2\pi]$,
(b) the Fourier series $\sum_{k,\ell=-\infty}^{+\infty} c_{k\ell} e^{ikx+i\ell y}$ on $[0, 2\pi]^2$.

Express the condition $u \in H^m$ in terms of the coefficients. In particular, show the equivalence of the assertions $u \in L_2$ and $c \in \ell_2$.

Show that in case (b), $u_{xx} + u_{yy} \in L^2$ implies $u_{xy} \in L^2$.

Solution. Let $v(x, y) = \sum_{k=-\infty}^{+\infty} c_k e^{ikx}$. The equivalence of $v \in L_2$ and $c \in \ell_2$ is a standard result of Fourier analysis. In particular,

$$
\begin{aligned}
v_x \in L_2 &\Leftrightarrow \sum_{k\ell} |k c_{k\ell}|^2 < \infty, \\
v_y \in L_2 &\Leftrightarrow \sum_{k\ell} |\ell c_{k\ell}|^2 < \infty, \\
v_{xx} \in L_2 &\Leftrightarrow \sum_{k\ell} |k^2 c_{k\ell}|^2 < \infty, \\
v_{xy} \in L_2 &\Leftrightarrow \sum_{k\ell} |k\ell c_{k\ell}|^2 < \infty, \\
v_{yy} \in L_2 &\Leftrightarrow \sum_{k\ell} |\ell^2 c_{k\ell}|^2 < \infty.
\end{aligned}
$$

If $v_{xx} + v_{yy} \in L_2$, then $\sum_{k\ell} |(k^2 + \ell^2) c_{k\ell}|^2 < \infty$. It follows immediately that v_{xx} and v_{yy} belong to L_2. Young's inequality $2|kl| \leq k^2 + \ell^2$ yields $\sum_{k\ell} |k\ell c_{k\ell}|^2 < \infty$ and $v_{xy} \in L_2$.

A simple regularity result for the solution of the Poisson equation on $[0, \pi]^2$ is obtained from these considerations. Let $f \in L_2([0, \pi]^2)$. We extend the domain to $[-\pi, \pi]^2$ by setting

$$
f(-x, y) = -f(x, y), \quad f(x, -y) = -f(x, y),
$$

and have an expansion

$$
f(x, y) = \sum_{k,\ell=1}^{\infty} c_{k\ell} \sin kx \sin \ell y.
$$

Since all the involved sums are absolutely convergent,

$$
u(x, y) = \sum_{k,\ell=1}^{\infty} \frac{c_{k\ell}}{k^2 + \ell^2} \sin kx \sin \ell y
$$

is a solution of $-\Delta u = f$ with homogeneous Dirichlet boundary conditions. The preceding equivalences yield $u \in H^2([0, \pi]^2)$. □

2.11 Let Ω be bounded with $\Gamma := \partial\Omega$, and let $g : \Gamma \to \mathbb{R}$ be a given function. Find the function $u \in H^1(\Omega)$ with minimal H^1-norm which coincides with g on Γ. Under what conditions on g can this problem be handled in the framework of this section?

Solution. Let g be the restriction of a function $u_1 \in C^1(\bar{\Omega})$. We look for $u \in H_0^1(\Omega)$ such that $\|u_1 + u\|_1$ is minimal. This variational problems is solved by

$$(\nabla u, \nabla v)_0 + (u, v)_0 = \langle \ell, v \rangle \quad \forall v \in H_0^1$$

with $\langle \ell, v \rangle := -(\nabla u_1, \nabla v)_0 - (u_1, v)_0$.

It is the topic of the next § to relax the conditions on the boundary values. □

2.12 Consider the elliptic, but not uniformly elliptic, bilinear form

$$a(u, v) := \int_0^1 x^2 u' v' \, dx$$

on the interval $[0, 1]$. Show that the problem $J(u) := \frac{1}{2} a(u, u) - \int_0^1 u dx \to$ min! does not have a solution in $H_0^1(0, 1)$. – What is the associated (ordinary) differential equation?

Solution. We start with the solution of the associated differential equation

$$-\frac{d}{dx} x^2 \frac{d}{dx} u = 1.$$

First we require only the boundary condition at the right end, i.e., $u(1) = 0$, and obtain with the free parameter A:

$$u(x) = -\log x + A(\frac{1}{x} - 1).$$

When we restrict ourselves to the subinterval $[\delta, 1]$ with $\delta > 0$ and require $u_\delta(\delta) = 0$, the (approximate) solution is

$$u_\delta(x) = -\log x + \frac{\delta \log \delta}{1 - \delta} (\frac{1}{x} - 1)$$

for $x > \delta$ and $u_\delta(x) = 0$ for $0 \le x \le \delta$. Note that $\lim_{\delta \to 0} u_\delta(x) = -\log x$ for each $x > 0$.

Elementary calculations show that $\lim_{\delta \to 0} J(u_\delta) = J(-\log x)$ and that $\|u_\delta\|_1$ is unbounded for $\delta \to 0$. There is no solution in $H_0^1(0, 1)$ although the functional J is bounded from below.

We emphasize another consequence. Due to Remark II.1.8 $H^1[a, b]$ is embedded into $C[a, b]$, but $\int_0^1 x^2 v'(x)^2 dx < \infty$ does not imply the continuity of v. □

2.14 In connection with Example 2.7, consider the continuous linear mapping

$$L : \ell_2 \to \ell_2,$$
$$(Lx)_k = 2^{-k} x_k.$$

Show that the range of L is not closed.
Hint: The closure contains the point $y \in \ell_2$ with $y_k = 2^{-k/2}$, $k = 1, 2, \ldots$.

Solution. Following the hint define the sequence $\{x^j\}$ in ℓ_2 by

$$x_k^j = \begin{cases} 2^{+k/2} & \text{if } j \leq k, \\ 0 & \text{otherwise.} \end{cases}$$

From $y = \lim_{j\to\infty} Lx^j$ it follows that y belongs to the closure of the range, but there is no $x \in \ell_2$ with $Lx = y$. □

3.7 Suppose the domain Ω has a piecewise smooth boundary, and let $u \in H^1(\Omega) \cap C(\bar{\Omega})$. Show that $u \in H_0^1(\Omega)$ is equivalent to $u = 0$ on $\partial\Omega$.

Solution. Instead of performing a calculation as in the proof of the trace theorem, we will apply the trace theorem directly.

Let $u \in H_0^1(\Omega) \cap C(\bar{\Omega})$ and suppose that $u(x_0) \neq 0$ for some $x_0 \in \Gamma$. There is a smooth part $\Gamma_1 \subset \Gamma$ with $x_0 \in \Gamma_1$ and $|u(x)| \geq \frac{1}{2}|u(x_0)|$ for $x \in \Gamma_1$. In particular, $\|u\|_{0,\Gamma_1} \neq 0$. By definition of $H_0^1(\Omega)$ there is a sequence $\{v_n\}$ in $C_0^\infty(\Omega)$ that converges to u. Clearly, $\|v_n\|_{0,\Gamma_1} = 0$ holds for all n, and $\lim_{n\to\infty} \|v_n\|_{0,\Gamma_1} = 0 \neq \|u\|_{0,\Gamma_1}$. This contradicts the continuity of the trace operator. We conclude from the contradiction that $u(x_0) = 0$. □

4.4 As usual, let u and u_h be the functions which minimize J over V and S_h, respectively. Show that u_h is also a solution of the minimum problem

$$a(u - v, u - v) \longrightarrow \min_{v \in S_h}!$$

Because of this, the mapping

$$R_h : V \longrightarrow S_h$$
$$u \longmapsto u_h$$

is called the *Ritz projector*.

Solution. Given $v_h \in S_h$, set $w_h := v_h - u_h$. From the Galerkin orthogonality (4.7) and the symmetry of the bilinear form we conclude with the Binomial formula that

$$\begin{aligned} a(u - v_h, u - v_h) &= a(u - u_h, u - u_h) + 2a(u - u_h, w_h) + a(w_h, w_h) \\ &= a(u - u_h, u - u_h) + a(w_h, w_h) \\ &\geq a(u - u_h, u - u_h). \end{aligned}$$

This proves that the minimum is attained at u_h. □

4.6 Suppose in Example 4.3 that on the bottom side of the square we replace the Dirichlet boundary condition by the natural boundary condition $\partial u/\partial \nu = 0$. Verify that this leads to the stencil

$$\begin{bmatrix} & -1 & \\ -1/2 & 2 & -1/2 \end{bmatrix}_*$$

at these boundary points.

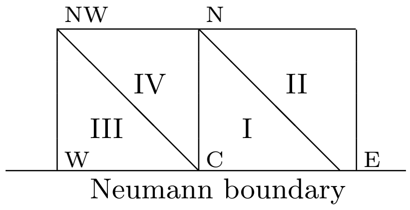

Fig. Numbering of the elements next to the center C on the Neumann boundary.

Solution. Let C be a point on the Neumann boundary. The boundary condition $\partial u/\partial \nu = 0$ is a natural boundary condition for the Poisson equation, and it is incorporated by testing u with the finite element functions in H^1 and not only in H_0^1. Specifically, it is tested with the nodal function ψ_C that lives on the triangles I–IV in the figure above. Recalling the computations in Example 4.3 we get

$$\begin{aligned} a(\psi_C, \psi_C) &= \int_{I-IV} (\nabla \psi_C)^2 dxdy \\ &= \int_{I+III+IV} [(\partial_1 \psi_C)^2 + (\partial_2 \psi_C)^2] dxdy \\ &= \int_{I+III} (\partial_1 \psi_C)^2 dxdy + \int_{I+IV} (\partial_2 \psi_C)^2 dxdy \\ &= h^{-2} \int_{I+III} dxdy + h^{-2} \int_{I+IV} dxdy = 2, \end{aligned}$$

There is no change in the evaluation of the bilinear form for the nodal function associated to the point north of C, i.e., $a(\psi_C, \psi_N) = -1$. Next we have

$$\begin{aligned} a(\psi_C, \psi_E) &= \int_I \nabla \psi_C \cdot \nabla \psi_E dxdy \\ &= \int_I \partial_1 \psi_C \partial_1 \psi_E dxdy = \int_I (-h^{-1}) h^{-1} dxdy = -1/2. \end{aligned}$$

Since the same number is obtained for $a(\psi_C, \psi_W)$, the stencil is as given in the problem. □

5.14 The completion of the space of vector-valued functions $C^\infty(\Omega)^n$ w.r.t. the norm

$$\|v\|^2 := \|v\|_{0,\Omega}^2 + \|\operatorname{div} v\|_{0,\Omega}^2$$

is denoted by $H(\operatorname{div},\Omega)$. Obviously, $H^1(\Omega)^n \subset H(\operatorname{div},\Omega) \subset L_2(\Omega)^n$. Show that a piecewise polynomial v is contained in $H(\operatorname{div},\Omega)$ if and only if the components $v \cdot \nu$ in the direction of the normals are continuous on the inter-element boundaries.
Hint: Apply Theorem 5.2 and use (2.22). — Similarly piecewise polynomials in the space $H(\operatorname{rot},\Omega)$ are characterized by the continuity of the tangential components; see Problem VI.4.8.

Solution. By definition, $w = \operatorname{div} v$ holds in the weak sense if

$$\int_\Omega w\phi dx = -\int_\Omega v \cdot \nabla\phi dx \quad \forall \phi \in C_0^\infty(\Omega). \tag{1}$$

Assume that $\Omega = \Omega_1 \cup \Omega_2$ and that $v|_{\Omega_i} \in C^1(\Omega_i)$ for $i = 1,2$. Set $\Gamma_{12} = \partial\Omega_1 \cap \partial\Omega_2$. By applying Green's formula to the subdomains we obtain

$$\begin{aligned} -\int_\Omega v \cdot \nabla\phi dx &= -\sum_{i=1}^2 \int_{\Omega_i} v \cdot \nabla\phi dx \\ &= \sum_{i=1}^2 \Big[\int_{\Omega_i} \operatorname{div} v\phi dx + \int_{\partial\Omega_i} v \cdot \phi\nu dx\Big] \\ &= \int_\Omega \operatorname{div} v\phi dx + \int_{\Gamma_{12}} [v\cdot\nu]\phi dx. \end{aligned} \tag{2}$$

Here $[\cdot]$ denotes the jump of a function. The right-hand side of (2) can coincide with the left-hand side of (1) for all $\phi \in C_0^\infty$ only if the jump of the normal component vanishes.

Conversely, if the jumps of the normal component vanish, then (1) holds if we set pointwise $w(x) := \operatorname{div} v(x)$, and this function is the divergence in the weak sense. □

6.12 Let $\mathcal{T}_h$ be a family of uniform partitions of Ω, and suppose S_h belong to an affine family of finite elements. Suppose the nodes of the basis are $z_1, z_2, \ldots, z_N$ with $N = N_h = \dim S_h$. Verify that for some constant c independent of h, the following inequality holds:

$$c^{-1}\|v\|_{0,\Omega}^2 \le h^2 \sum_{i=1}^N |v(z_i)|^2 \le c\|v\|_{0,\Omega}^2 \quad \text{for all } v \in S_h.$$

Solution. Let $\hat{z}_1, \hat{z}_2, \ldots, \hat{z}_s$ be the nodes of a basis of the s-dimensional space Π on the reference triangle T_{ref}. The norm

$$|||v||| := \Big(\sum_{i=1}^{s} |v(\hat{z}_i)|^2\Big)^{1/2}$$

is equivalent to $\|\cdot\|_{0,T_{\text{ref}}}$ on Π since Π is a finite dimensional space. Let T_h be an element of $\mathcal{T}_h$ with diameter h. A scaling argument in the spirit of the transformation formula 6.6 shows that

$$\|v\|_{0,T_h} \quad \text{and} \quad h^2 \sum_{z_i \in T_h} |v(z_i)|^2$$

differ only by a factor that is independent of h. By summing over all elements of the triangulation we obtain the required formula. □

6.13 Under appropriate assumptions on the boundary of Ω, we showed that

$$\inf_{v \in S_h} \|u - v_h\|_{1,\Omega} \le c\, h \|u\|_{2,\Omega}\ ,$$

where for every $h > 0$, S_h is a finite-dimensional finite element space. Show that this implies the compactness of the imbedding $H^2(\Omega) \hookrightarrow H^1(\Omega)$. [Thus, the use of the compactness in the proof of the approximation theorem was not just a coincidence.]

Solution. Let B be the unit ball in $H^2(\Omega)$.

Let $\varepsilon > 0$. Choose h such that $ch < \varepsilon/4$, and for any $u \in B$ we find $v_h \in S_h$ with $\|u - v_n\|_1 \le \varepsilon/4$. Since $\dim S_h$ is finite, the bounded set $\{v \in S_h; \|v\|_1 \le 1\}$ can be covered by a finite number of balls with diameter $\varepsilon/2$. If the diameter of these balls are doubled, they cover the set B. Hence, B is precompact, and the completeness of the Sobolev space implies compactness. □

6.14 Let $\mathcal{T}_h$ be a κ-regular partition of Ω into parallelograms, and let u_h be an associated bilinear element. Divide each parallelogram into two triangles, and let $\|\cdot\|_{m,h}$ be defined as in (6.1). Show that

$$\inf \|u_h - v_h\|_{m,\Omega} \le c(\kappa) h^{2-m} \|u_h\|_{2,\Omega}, \quad m = 0, 1,$$

where the infimum is taken over all piecewise linear functions on the triangles in $\mathcal{M}^1$.

Solution. The combination of the idea of the Bramble–Hilbert–Lemma and a scaling argument is typical for a priori error estimates.

Given a parallelogram $T_j \in \mathcal{T}_h$ the interpolation operator

$$\begin{aligned} I : H^2(T_J) &\to \mathcal{M}^1|_{T_j} \\ (Iu)(z_i) &= u(z_i) \;\forall \text{ nodes } z_i \text{ of } T_j \end{aligned}$$

is bounded

$$\|Iu\|_{1,T_j} \le c(\kappa)\|u\|_{2,T_j}.$$

Since $Iu = u$ if u is a linear polynomial, we conclude from Lemma 6.2 that

$$\|u - Iu\|_{1,T_j} \le c(\kappa)|u|_{2,T_j}.$$

The standard scaling argument shows that

$$\|u - Iu\|_{m,T_j} \le c(\kappa)h^{2-m}|u|_{2,T_j} \quad m = 0, 1.$$

The extension to the domain Ω is straight forward. After setting $v_h = Iu_h$ and summing the squares over all parallelograms in $\mathcal{T}_h$ the proof is complete. □

7.11 Let $\Omega = (0, 2\pi)^2$ be a square, and suppose $u \in H_0^1(\Omega)$ is a weak solution of $-\Delta u = f$ with $f \in L_2(\Omega)$. Using Problem 1.16, show that $\Delta u \in L_2(\Omega)$, and then use the Cauchy–Schwarz inequality to show that all second derivatives lie in L_2, and thus u is an H^2 function.

Solution. We rather let $\Omega = (0, \pi)^2$ since this does not change the character of the problem.

We extend f to $\Omega_{sym} := (-\pi, \pi)^2$ by the (anti-) symmetry requirements

$$f(-x, y) = -f(x, y), \quad f(x, -y) = -f(x, y),$$

without changing the symbol. Since $f \in L_2(\Omega_{sym})$, f can be represented as a Fourier series with sine functions only

$$f(x, y) = \sum_{k,\ell=1}^{\infty} a_{k\ell} \sin kx \sin \ell y.$$

Parseval's inequality yields

$$\sum_{k,\ell} |a_{k\ell}|^2 = \pi^2 \|f\|_{2,\Omega}.$$

Obviously, the solution has the representation

$$u(x,y) = \sum_{k\ell} \frac{a_{k\ell}}{k^2+\ell^2} \sin kx \sin \ell y.$$

The coefficients in the representation

$$u_{xx} = -\sum_{k,\ell} \frac{k^2}{k^+\ell^2} a_{k\ell} \sin kx \sin \ell y$$

are obviously square summable, and $u_{xx} \in L_2(\Omega)$. The same is true for u_{yy}. More interesting is

$$u_{xy} = \sum_{k,\ell} \frac{k\ell}{k^2+\ell^2} a_{k\ell} \cos kx \cos \ell y.$$

From Young's inequality $2k\ell \le k^2+\ell^2$ we conclude that we have square summability also here. Hence, $u_{xy} \in L_2(\Omega)$, and the proof of $u \in H^2(\Omega)$ is complete. □

Chapter III

1.11 If the stiffness matrices are computed by using numerical quadrature, then only approximations a_h of the bilinear form are obtained. This holds also for conforming elements. Estimate the influence on the error of the finite element solution, given the estimate

$$|a(u,v) - a_h(u,v)| \le \varepsilon(h)\, \|u\|_1 \|v\|_1 \quad \text{for all } v \in S_h.$$

Moreover, assume that the two bilinear forms are coercive with the constant $\alpha > 0$.

Note that the original assumption in the book has to be replaced by the more restrictive assumption above, since the difference $a(.,.) - a_h(.,.)$ need not be coercive.

Solution. Let u_h and w_h be the solutions of

$$a(u_h, v) = (f, v) \quad \forall v \in S_h,$$
$$a_h(w_h, v) = (f, v) \quad \forall v \in S_h,$$

Hence, $a(u_h - w_h, v) = a_h(w_h, v) - a(w_h, v)$, and by setting $v := u_h - w_h$ we obtain

$$\alpha \|u_h - w_h\|_1^2 \le a(u_h - w_h, u_h - w_h) \le \varepsilon(h)\, \|w_h\|_1 \|u_h - w_h\|_1.$$

Now we divide by $\alpha\|u_h - w_h\|_1$, note that $a(w_h, w_h) = (f, w_h)$, and recall the coercivity of the bilinear forms to obtain

$$\|u_h - w_h\|_1 \le \varepsilon(h)\, \alpha^{-2} \|f\|.$$

We have to add this term to the standard error estimate for $\|u - u_h\|_1$. □

1.12 The Crouzeix–Raviart element has locally the same degrees of freedom as the conforming P_1 element $\mathcal{M}_0^1$, i. e. the Courant triangle. Show that the (global) dimension of the finite element spaces differ by a factor that is close to 3 if a rectangular domain as in Fig. 9 is partitioned.

Solution. The nodal variables of the conforming P_1 element are associated to the nodes of a mesh (as in Fig. 9) with mesh size h.

The nodal points of the corresponding nonconforming P_1 element are associated to the mesh with meshsize $h/2$, but with those of the h-mesh excluded. Since halving the meshsize induces a factor of about 4 in the number of points, the elimination of the original points gives rise to a factor of about 3. □

3.8 Let $a : V \times V \to \mathbb{R}$ be a positive symmetric bilinear form satisfying the hypotheses of Theorem 3.6. Show that a is elliptic, i.e., $a(v,v) \ge \alpha_1 \|v\|_V^2$ for some $\alpha_1 > 0$.

Solution. Given u, by the inf-sup condition there is a $v \ne 0$ such that $\frac{1}{2}\alpha\|u\|_V \le a(u,v)/\|v\|_V$. The Cauchy inequality and (3.6) yield

$$\frac{1}{4}\alpha^2\|u_h\|_V^2 \le \frac{a(u,v)^2}{\|v\|_V^2} \le a(u,u)\frac{a(v,v)}{\|v\|_V^2} \le Ca(u,u).$$

Therefore, we have ellipticity with $\alpha_1 \ge \alpha^2/(4C)$.

3.9 [Nitsche, private communication] Show the following converse of Lemma 3.7: Suppose that for every $f \in V'$, the solution of (3.5) satisfies

$$\lim_{h\to 0} u_h = u := L^{-1}f.$$

Then

$$\inf_h \inf_{u_h\in U_h} \sup_{v_h\in V_h} \frac{a(u_h,v_h)}{\|u_h\|_U\|v_h\|_V} > 0.$$

Hint: Use (3.10) and apply the principle of uniform boundedness.

Solution. Given $f \in V'$, denote the solution of (3.5) by u_h. Let $K_h : V' \to U_h \subset U$ be the mapping that sends f to u_h. Obviously, K_h is linear. To be precise, we assume that u_h is always well defined. Since $\|f|_{V_h'}\|_{V'} \le \|f\|_{V'}$, each K_h is a bounded linear mapping. From $\lim_{h\to 0} K_h f = L^{-1}f$ we conclude that $\sup_h \|K_h f\| < \infty$ for each $f \in V'$. The principle of uniform boundedness assures that

$$\alpha^{-1} := \sup_h \|K_h\| < \infty.$$

Hence, $\|K_h u_h\| \ge \alpha\|u_h\|$ holds for each $u_h \in V'$. Finally, the equivalence of (3.7) and (3.10) yields the inf-sup condition with the uniform bound $\alpha > 0$. □

3.10 Show that

$$\begin{aligned} \|v\|_0^2 &\le \|v\|_m \|v\|_{-m} \quad \text{for all } v \in H_0^m(\Omega), \\ \|v\|_1^2 &\le \|v\|_0 \|v\|_2 \quad \text{for all } v \in H^2(\Omega) \cap H_0^1(\Omega). \end{aligned}$$

Hint: To prove the second relation, use the Helmholtz equation $-\Delta u + u = f$.

Solution. By definition II.3.1 we have

$$(u, v)_0 \le \|u\|_{-m} \|v\|_m.$$

Setting $u := v$ we obtain $\|v\|_0^2 \le \|v\|_{-m} \|v\|_m$, i.e., the first statement.

Since zero boundary conditions are assumed, Green's formula yields

$$\int_\Omega w_i \partial_i v ds = - \int_\Omega \partial_i w_i v dx.$$

Setting $w_i := \partial_i v$ and summing over i we obtain

$$\int_\Omega \nabla v \cdot \nabla v dx = - \int_\Omega \Delta v \, v dx.$$

With the Cauchy inequality and $\|\nabla v\|_0 \le \|v\|_2$ the inequality for $s = 1$ is complete. □

3.12 (Fredholm Alternative) Let H be a Hilbert space. Assume that the linear mapping $A : H \to H'$ can be decomposed in the form $A = A_0 + K$, where A_0 is H-elliptic, and K is compact. Show that either A satisfies the inf-sup condition, or there exists an element $x \in H$, $x \ne 0$, with $Ax = 0$.

Solution. If A does not satisfy an inf-sup condition, there is a sequence $\{x_n\}$ with $\|x_n\| = 1$ and $Ax_n \to 0$. Since K is compact, a subsequence of $\{Kx_n\}$ converges. Without loss of generality we may assume that $\lim_{n\to\infty} Kx_n = q$, $q \in H'$. It follows that

$$\lim_{n\to\infty} A_0 x_n = \lim_{n\to\infty} Ax_n - \lim_{n\to\infty} Kx_n = 0 - q = -q.$$

Since A_0 is invertible, the sequence $\{x_n\}$ converges to $z := -A_0^{-1} q$, and $Az = \lim_{n\to\infty} A_0 x_n + \lim_{n\to\infty} K x_n = 0$. Moreover, $\|z\| = 1$. □

4.16 Show that the inf-sup condition (4.8) is equivalent to the following decomposition property: For every $u \in X$ there exists a decomposition

$$u = v + w$$

with $v \in V$ and $w \in V^\perp$ such that

$$\|w\|_X \le \beta^{-1} \|Bu\|_{M'},$$

where $\beta > 0$ is a constant independent of u.

Solution. This problem is strongly related to Lemma 4.2(ii). Assume that (4.8) holds. Given $u \in X$, since V and $V^\perp$ are closed, there exists an orthogonal decomposition

$$u = v + w, \quad v \in V, w \in V^\perp. \tag{1}$$

From Lemma 4.2(ii) it follows that $\|Bw\|_{M'} \ge \beta \|w\|_X$. Since v in the decomposition (1) lies in the kernel of B, we have $\|w\|_X \le \beta^{-1}\|Bw\|_{M'} = \beta^{-1}\|Bu\|_{M'}$.

Conversely, assume that the decomposition satisfies the conditions as formulated in the problem. If $u \in V^\perp$, then $v = 0$ and $\|u\|_X \le \beta^{-1}\|Bu\|_{M'}$ or $\|Bu\|_{M'} \ge \beta\|u\|_X$. Hence, the statement in Lemma 4.2(ii) is verified. □

4.21 The pure Neumann Problem (II.3.8)

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} &= g \quad \text{on } \partial\Omega \end{aligned}$$

is only solvable if $\int_\Omega f\,dx + \int_\Gamma g\,ds = 0$. This compatibility condition follows by applying Gauss' integral theorem to the vector field ∇u. Since u+const is a solution whenever u is, we can enforce the constraint

$$\int_\Omega u dx = 0.$$

Formulate the associated saddle point problem, and use the trace theorem and the second Poincaré inequality to show that the hypotheses of Theorem 4.3 are satisfied.

Solution. Consider the saddle-point problem with $X = H^1(\Omega)$, $M = \mathbb{R}$, and the bilinear forms

$$\begin{aligned} a(u,v) &= \int_\Omega \nabla u \nabla v dx, \\ b(u,\lambda) &= \lambda \int_\Omega v dx = \lambda \bar{v} \mu(\Omega). \end{aligned}$$

Adopt the notation of Problem II.1.12. With the variant of Friedrich's inequality there we obtain

$$\begin{aligned}\|v\|_1^2 &= |v|_1^2 + \|v\|_0^2 \le |v|_1^2 + 2c^2\big(|\bar v|^2 + |v|_1^2\big)\\ &\le c^1[a(v,v) + |\bar v|]^2\\ &= c^1 a(v,v) \qquad \text{if } \bar v = 0.\end{aligned}$$

This proves ellipticity of $a(\cdot,\cdot)$ on the kernel.

The inf-sup condition is verified by taking the constant test function $v_0 = 1$:

$$b(\lambda, v_0) = \lambda \int_\Omega dx = \lambda\mu(\Omega) = \lambda\|v_0\|_0\mu(\Omega)^{1/2} = \lambda\|v_0\|_1\mu(\Omega)^{1/2}.$$

The condition holds with the constant $\mu(\Omega)^{1/2}$. □

4.22 Let a, b, and c be positive numbers. Show that $a \le b + c$ implies that $a \le b^2/a + 2c$.

Solution.

$$a \le b(b+c)/(b+c) + c = b^2/(b+c) + c(1 + b/(b+c)) \le b^2/a + 2c. \quad □$$

6.8 [6.7 in 2nd ed.] Find a Stokes problem with a suitable right-hand side to show the following: Given $g \in L_{2,0}(\Omega)$, there exists $u \in H_0^1(\Omega)$ with

$$\operatorname{div} u = g \quad \text{and} \quad \|u\|_1 \le c\|g\|_0\,,$$

where as usual, c is a constant independent of q. [This means that the statement in Theorem 6.3 is also necessary for the stability of the Stokes problem.]

Solution. We consider the saddle.point with the same bilinear forms as in (6.5), but with different right -hand sides,

$$\begin{aligned} a(u,v) + b(v,p) &= 0 && \text{for all } v \in X,\\ (\operatorname{div} u, q)_0 &= (g,q)_0 && \text{for all } q \in M.\end{aligned}$$

The inf-sup condition guarantees the existence of a solution $u \in H_0^1(\Omega)$ with $\|u\|_1 \le c\|g\|_0$. The zero boundary conditions imply $\int_{\partial\Omega} u\,\nu\,ds = 0$, and it follows from the divergence theorem that $\int_\Omega g\,dx = 0$. Hence, both $\operatorname{div} u$ and g live in $M = L_{2,0}$. Now, the second variational equality implies that the two functions are equal. □

Note. The consistency condition $\int_\Omega g\,dx = 0$ was missing in the second English edition, and there is only a solution $u \in H^1(\Omega)$. The addition of a multiple of the linear function $u_1 = x_1$ yields here the solution. – We have changed the symbol for the right-hand side in order to have a consistent notation with (6.5).

6.8 [7.4 in 2nd ed.] If Ω is convex or sufficiently smooth, then one has for the Stokes problem the regularity result

$$\|u\|_2 + \|p\|_1 \le c\|f\|_0; \tag{7.18}$$

see Girault and Raviart [1986]. Show by a duality argument the L_2 error estimate

$$\|u - u_h\|_0 \le ch(\|u - u_h\|_1 + \|p - p_h\|_0). \tag{7.19}$$

Solution. As usually in duality arguments consider an auxiliary problem. *Find* $\varphi \in X$, $r \in M$ *such that*

$$\begin{aligned} a(w, \varphi) + b(w, r) &= (u - u_0, w)_0 \quad \text{for all } w \in X, \\ b(\varphi, q) \qquad\qquad &= 0 \qquad\qquad\quad \text{for all } q \in M. \end{aligned} \tag{1}$$

The regularity assumption yields $\|\varphi\|_2 + \|r\|_1 \le C\|u - u_0\|_0$, and by the usual approximation argument there are $\varphi_h \in X_h$, $r_h \in M_h$ such that

$$\|\varphi - \varphi_h\|_1 + \|r - r_h\|_0 \le Ch\|u - u_0\|_0.$$

The subtraction of (4.4) and (4.5) with the test function φ_h, r_h yields the analogon to Galerkin orthogonality

$$\begin{aligned} a(u - u_h, \varphi_h) + b(\varphi_h, p - p_h) &= 0, \\ b(u - u_h, r_h) \qquad\qquad\qquad &= 0. \end{aligned}$$

Now we set $w := u - u_h$, $q := p - p_h$ in (1) and obtain

$$\begin{aligned} &(u - u_h, u - u_h)_0 = a(u - u_h, \varphi) + b(u - u_h, r) + b(\varphi, p - p_h) \\ &\quad = a(u - u_h, \varphi - \varphi_h) + b(u - u_h, r - r_h) + b(\varphi - \varphi_h, p - p_h) \\ &\quad \le C(\|u - u_h\|_1\|\varphi - \varphi_h\|_1 + \|u - u_h\|_1\|r - r_h\|_1 + \|\varphi - \varphi_h\|_1\|p - p_h\|_0) \\ &\quad \le C(\|u - u_h\|_1 + \|u - u_h\|_1 + \|p - p_h\|_0)\, h\|u - u_h\|_0\,. \end{aligned}$$

After dividing by $\|u - u_h\|_0$ the proof is complete. □

9.6 Consider the Helmholtz equation

$$\begin{aligned} -\Delta u + \alpha u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

with $\alpha > 0$. Let $v \in H_0^1(\Omega)$ and $\sigma \in H(\text{div}, \Omega)$ satisfy $\text{div}\,\sigma + f = \alpha v$.

Show the inequality of Prager–Synge type with a computable bound

$$\begin{aligned}|u-v|_1^2+\alpha\|u-v\|_0^2 \\ +\|\operatorname{grad} u-\sigma\|_0^2+\alpha\|u-v\|_0^2=\|\operatorname{grad} v-\sigma\|_0^2\,.\end{aligned} \tag{9.11}$$

Recall the energy norm for the Helmholtz equation in order to interpret (9.13).

Solution. First we apply the Binomial formula

$$\begin{aligned}\|\operatorname{grad} v-\sigma\|_0^2 &= \|\operatorname{grad}(v-u)-(\sigma-\operatorname{grad} u)i\|_0^2 \\ &= \|\operatorname{grad}(v-u)i\|_0^2+\|\sigma-\operatorname{grad} ui\|_0^2 \\ &\quad -2\int_\Omega \operatorname{grad}(v-u)(\sigma-\operatorname{grad} u)dx\end{aligned}$$

.

Green's formula yields an expression with vanishing boundary integral

$$\begin{aligned}-\int_\Omega \operatorname{grad}(v-u)(\sigma-\operatorname{grad} u)dx &= \int_\Omega (v-u)(\operatorname{div}\sigma-\Delta u)dx \\ &\quad +\int_{\partial\Omega}(v-u)\Big(\sigma\cdot n-\frac{\partial u}{dn}\Big)ds \\ &= \int_\Omega (v-u)[-f+\alpha v+f-\alpha v]dx+0 \\ &= \int_\Omega \alpha(v-u)^2dx=\alpha\|v-u\|_0^2.\end{aligned}$$

By collecting terms we obtain (9.11).

Note that $\sqrt{\|\operatorname{grad}(v)i\|_0^2+\alpha\|v\|_0^2}$ is here the energy norm of v. □

Chapter IV

2.6 By (2.5), $\alpha_k \ge \alpha^* := 1/\lambda_{\max}(A)$. Show that convergence is guaranteed for every fixed step size α with $0<\alpha<2\alpha^*$.

Solution. We perform a spectral decomposition of the error

$$x_k-x^*=\sum_{j=1}^n \beta_j z_j$$

with $Az_j=\lambda_j z_j$ for $j=1,\dots,n$. The iteration

$$x_{k+1}=x_k+\alpha(b-Ax_k)$$

leads to

$$x_{k+1} - x^* = (1 - \alpha A)(x_k - x^*) = \sum_{j=1}^{n} (1 - \alpha\lambda_j)\beta_j z_j.$$

The damping factors satisfy $-1 < 1 - \alpha\lambda_j < 1$ if $0 < \alpha < 2/\lambda_{\max}(A)$, and convergence is guaranteed. □

4.8 Show that the matrix

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

is positive definite, and that its condition number is 4.

Hint: The quadratic form associated with the matrix A is $x^2 + y^2 + z^2 + (x + y + z)^2$.

Solution. The formula in the hint shows that $A \geq I$. By applying Young's inequality to the nondiagonal terms, we see that $(x+y+z)^2 \leq 3(x^2+y^2+z^2)$ and $A \leq 4I$. The quotient of the factors in the upper and the lower bound is 4. □

4.14 Let $A \leq B$ denote that $B - A$ is positive semidefinite. Show that $A \leq B$ implies $B^{-1} \leq A^{-1}$, but it does not imply $A^2 \leq B^2$. — To prove the first part note that $(x, B^{-1}x) = (A^{-1/2}x, A^{1/2}B^{-1}x)$ and apply Cauchy's inequality. Next consider the matrices

$$A := \begin{pmatrix} 1 & a \\ a & 2a^2 \end{pmatrix} \quad \text{and} \quad B := \begin{pmatrix} 2 & 0 \\ 0 & 3a^2 \end{pmatrix}$$

for establishing the negative result. From the latter it follows that we cannot derive good preconditioners for the biharmonic equation by applying those for the poisson equation twice.

Note: The converse is more favorable, i.e., $A^2 \leq B^2$ implies $A \leq B$. Indeed, the Rayleigh quotient $\lambda = \max\{(x, Ax)/(x, Bx)$ is an eigenvalue, and the maximum is attained at an eigenvector, i.e., $Ax = \lambda Bx$. On the other hand, by assumption

$$0 \leq (x, B^2x) - (x, A^2x) = (1 - \lambda^2)\, \|Bx\|^2.$$

Hence, $\lambda \leq 1$ and the proof of the note is complete.

Solution. By Cauchy's inequality and $A \le B$ it follows that

$$\begin{aligned}(x, B^{-1}x)^2 = (A^{-1/2}x, A^{1/2}B^{-1}x)^2 &\le (x, A^{-1}x)\ (B^{-1}x, AB^{-1}x) \\ &\le (x, A^{-1}x)\ (B^{-1}x, BB^{-1}x).\end{aligned}$$

We divide by $(x, B^{-1}x)$ and obtain $B^{-1} \le A^{-1}$.

Consider the given matrices. The relation $(x, Ax) \le (x, Bx)$ is established by applying Young's inequality to the nondiagonal terms. Furthermore

$$A^2 = \begin{pmatrix} 1+a^2 & a+2a^3 \\ a+2a^3 & a^2+4a^4 \end{pmatrix}, \quad B^2 = \begin{pmatrix} 4 & 0 \\ 0 & 9a^4 \end{pmatrix}.$$

Obviously $B^2 - A^2$ has a negative diagonal entry if $a \ge 2$. □

4.15 Show that $A \le B$ implies $B^{-1}AB^{-1} \le B^{-1}$.

Solution. If $x = B^{-1}z$, then $(x, Ax) \le (x, Bx)$ reads
$(B^{-1}z, AB^{-1}z) \le (B^{-1}z, BB^{-1}z)$, i.e., $(z, B^{-1}AB^{-1}z) \le (z, B^{-1}z)$. □

4.16 Let A and B be symmetric positive definite matrices with $A \le B$. Show that

$$(I - B^{-1}A)^m B^{-1}$$

is positive definite for $m = 1, 2, \ldots$. To this end note that

$$q(XY)X = Xq(YX)$$

holds for any matrices X and Y if q is a polynomial. Which assumption may be relaxed if m is even?

Remark: We can only show that the matrix is semidefinite since $A = B$ is submitted by the assumptions.

Solution. First let m be an even number, $m = 2n$. We compute

$$\begin{aligned}(x, (I - B^{-1}A)^{2n}B^{-1}x) &= (x, (I - B^{-1}A)^n B^{-1}(I - AB^{-1})^n x) \\ &= ((I - AB^{-1})^n x, B^{-1}(I - AB^{-1})x) \\ &= (z, B^{-1}z) \ge 0,\end{aligned}$$

where $z := (I - AB^{-1})^n x$. This proves that the matrix is positive semidefinite. [Here we have only used that B is invertible.]

Similar we get with z as above

$$\begin{aligned}(x, (I - B^{-1}A)^{2n+1}B^{-1}x) &= (z, B^{-1}(I - AB^{-1})z) \\ &= (z, (B^{-1} - B^{-1}AB^{-1})z).\end{aligned}$$

The preceding problem made clear that $B^{-1} - B^{-1}AB^{-1} \ge 0$. □

Chapter V

2.11 Show that for the scale of the Sobolev spaces, the analog

$$\|v\|_{s,\Omega}^2 \le \|v\|_{s-1,\Omega}\,\|v\|_{s+1,\Omega}$$

of (2.5) holds for $s = 0$ and $s = 1$.

For the solution look at Problem III.3.10. □

5.7 Let V, W be subspaces of a Hilbert space H. Denote the projectors onto V and W by P_V, P_W, respectively. Show that the following properties are equivalent:
(1) A strengthened Cauchy inequality (5.3) holds with $\gamma < 1$.
(2) $\|P_W v\| \le \gamma\|v\|$ holds for all $v \in V$.
(3) $\|P_V w\| \le \gamma\|w\|$ holds for all $w \in W$.
(4) $\|v + w\| \ge \sqrt{1-\gamma^2}\,\|v\|$ holds for all $v \in V, w \in W$.
(5) $\|v + w\| \ge \sqrt{\frac{1}{2}(1-\gamma)}\,(\|v\| + \|w\|)$ holds for all $v \in V, w \in W$.

Solution. We restrict ourselves on the essential items.
(1) ⇒(2). Assume that the strengthened Cauchy inequality holds. Let $v \in V$ and $w_0 = P_W v$. It follows from the definition of the projector and the strengthened Cauchy inequality that

$$(w_0, w_0) = (w_0) \le \gamma\|v\|\,\|w_0\|.$$

After dividing by $\|w_0\|$ we obtain the property (2).
(2) ⇒(1). Given nonzero vectors $v \in V$ and $w \in W$, set $\alpha = |(v, w)|/\|v\|\,\|w\|$. Denote the closest point on span$\{w\}$ to v by w_0. It follows by the preceding item that $\|w\| = \alpha\|v\|$. By the orthogonality relations for nearest points we have

$$\begin{aligned}\gamma^2\|v\|^2 &\ge \|P_w\|^2 = \|v\|^2 - \|v - P_W v\|^2\\ &\ge \|v\|^2 - \|v - w_0\|^2 = \|w_0\|^2 = \alpha^2\|v\|^2.\end{aligned}$$

Hence, $\alpha \le \gamma$, and the strengthened Cauchy inequality is true.
(1) ⇒(4). It follows from the strengthened Cauchy inequality that

$$\begin{aligned}\|v + w\|^2 &= \|v\|^2 + 2(v, w) + \|w\|^2\\ &\ge \|v\|^2 - 2\gamma\|v\|\,\|w\| + \|w\|^2 = (1-\gamma^2)\|v\|^2 + (\gamma\|v\| - \|w\|)^2\\ &\ge (1-\gamma^2)\|v\|^2,\end{aligned}$$

and property (4) is true.
(1) ⇒(5). The strengthened Cauchy inequality implies

$$\begin{aligned}\|v+w\|^2 &\geq \|v\|^2 - \gamma(v,w) + \|w\|^2 \\ &= \frac{1}{2}(1-\gamma)(\|v\|+\|w\|)^2 + \frac{1}{2}(1+\gamma)(\|v\|-\|w\|)^2 \\ &\geq \frac{1}{2}(1-\gamma)(\|v\|+\|w\|)^2\end{aligned}$$

This proves property (5).
(5) ⇒(1). By assumption

$$\begin{aligned}2(v,w) &= \|v\|^2 + \|w\|^2 - \|v-w\|^2 \\ &\leq \|v\|^2 + \|w\|^2 - \frac{1}{2}(1-\gamma)(\|v\|+\|w\|)^2\end{aligned}$$

Since the relation is homogeneous, it is sufficient to verify the assertion for the case $\|v\| = \|w\| = 1$. Here the preceding inequality yields

$$2(v,w) \leq 1 + 1 - 2(1-\gamma) = 2\gamma = 2\gamma\|v\|\,\|w\|,$$

and the strengthened Cauchy inequality holds. □

Chapter VI

6.11 Show that

$$\|\operatorname{div}\eta\|_{-1} \leq \text{ const } \sup_\gamma \frac{(\gamma,\eta)_0}{\|\gamma\|_{H(\mathrm{rot},\Omega)}},$$

and thus that $\operatorname{div}\eta \in H^{-1}(\Omega)$ for $\eta \in (H_0(\mathrm{rot},\Omega))'$. Since $H_0(\mathrm{rot},\Omega) \supset H_0^1(\Omega)$ implies $(H_0(\mathrm{rot},\Omega))' \subset H^{-1}(\Omega)$, this completes the proof of (6.9).

Solution. Let $v \in H_0^1(\Omega)$. Its gradient $\gamma := \nabla v$ satisfies $\nabla v \cdot \tau = 0$ on $\partial\Omega$. Since $\mathrm{rot}\,\nabla v = 0$, we have $\gamma \in H_0(\mathrm{rot},\Omega)$ and $\|\gamma\|_0 = \|\gamma\|_{H_0(\mathrm{rot},\Omega)}$. Partial integration yields

$$\begin{aligned}\|\operatorname{div}\eta\|_{-1} &= \sup_{v\in H_0^1(\Omega)} \frac{(v,\operatorname{div}\eta)_0}{\|v\|_1} \\ &= \sup_{v\in H_0^1(\Omega)} \frac{(\nabla v,\eta)_0}{(\|\nabla v\|_0^2 + \|v\|_0^2)^{1/2}} \\ &\leq \sup_\gamma \frac{(\gamma,\eta)_0}{\|\gamma\|_{H_0(\mathrm{rot},\Omega)}}\,.\end{aligned}$$

A standard density argument yields $\operatorname{div}\eta \in H^{-1}(\Omega)$. □